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## OPTIMAL CONTROLS FOR STOCHASTIC FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

M. A. DIOP, P. D. A. GUINDO, M. FALL, AND A. DIAKHABY

**ABSTRACT.** The aim of this work is to investigate a class of stochastic functional integrodifferential equations(SFIDEs) in a Hilbert space. We first study the existence of mild solutions of these equations by means of stochastic analysis theory and theory of resolvent operator in the sense of Grimmer. Further, the existence of optimal pairs for the corresponding Lagrange control systems is investigated. Finally, an example is presented to illustrate our obtained results.

### 1. INTRODUCTION

In the last decades stochastic differential equations have attracted considerable attention. These equations have been studied extensively since they are abstract formulations for many problems arising from economics, finance, physics, mechanics, electricity and control engineering, etc. (see [10, 15, 25]). There is much current interest in studying qualitative properties for SPDEs (see, e.g., [1, 2, 5, 26, 27]). In recent years, much attention has been paid to the qualitative properties of mild solutions to various stochastic integrodifferential equations by using the resolvent operator theory for integral equations and the fixed point technique see e.g., [14, 20, 28] and the references therein.

On the other hand the optimal control is one of the important concepts in control theory and plays a vital role in control systems. For an optimal control problem, the minimization of a criterion function of the states and control inputs of the system over a set of admissible control functions are necessary. The system is subject to constrained dynamics and control variables, among which additional constraints such as final time constraints can be considered. The optimal control theory has been successfully applied in biology, engineering, economy, physics, etc. (see [12]). In recent years, many efforts have been made to investigate the existence of optimal controls for various types of stochastic nonlinear functional differential equations in infinite dimensional spaces(see [21, 25]). Bonaccorsi et al. [17] investigated the optimal control problem for stochastic differential equations with dynamical boundary conditions. Zhou and Liu [11] studied the existence of

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optimal control for stochastic evolution equations in Hilbert spaces. Ren and Wu [22] discussed the optimal control problem associated with multivalued stochastic differential equations with Levy jumps by using Yosida approximation theory. Rajivganthi et al. [18] presented the optimal control results of fractional stochastic neutral differential equations in Hilbert spaces. Very recently in [6], the authors obtained necessary conditions of optimality for some stochastic integrodifferential of neutral type. Dhayal et al. [13] studied the optimal control problem for non-instantaneous impulsive stochastic neutral integrodifferential equations driven by fractional Brownian motion by using the theory of the resolvent operator and a fixed point technique.

Motivated by the above discussion, in this work, we study the stochastic functional integrodifferential equations of the following form

$$\begin{cases} dx(t) = \left[ Ax(t) + \int_0^t B(t-s)x(s)ds + C(t)u(t) \right. \\ \quad \left. + \sigma(t, x_t) \right] dt + f(t, x_t)d\omega(t) \text{ for } t \in [0, T], \\ x(0) = \varphi \in \mathcal{B}, \end{cases} \quad (1)$$

where the state  $x(t)$  takes values in a separable real Hilbert space  $\mathbb{H}$  with inner product  $\langle \cdot, \cdot \rangle_{\mathbb{H}}$  and norm  $\|\cdot\|_{\mathbb{H}}$ .  $A$  is the infinitesimal generator of  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  on  $\mathbb{H}$  with domain  $D(A)$ . Let  $\mathbb{K}$  be another separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle_{\mathbb{K}}$  and norm  $\|\cdot\|_{\mathbb{K}}$ . Here  $(B(t))_{t \geq 0}$  is closed linear operator on  $\mathbb{H}$  with domain  $D(B) \subset D(A)$  which is independent of  $t$ . Suppose  $\{\omega(t); t \geq 0\}$  is a given  $\mathbb{K}$ -valued Wiener process with covariance operator  $Q > 0$  defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ , where  $\{\mathcal{F}_t\}_{t \geq 0}$  is a normal filtration generated by the Wiener process  $\omega$ .  $u$  takes values from separable, reflexive Hilbert space  $\mathbb{Y}$  and  $C$  is a linear operator from  $\mathbb{Y}$  into  $\mathbb{H}$ . The time history  $x_t : (-\infty, 0] \rightarrow \mathbb{H}$  given by  $x_t(\theta) = x(t + \theta)$  belongs to some abstract phase space  $\mathcal{B}$  defined axiomatically;  $f : [0, T] \times \mathcal{B} \rightarrow \mathcal{L}_b(\mathbb{K}, \mathbb{H})$  and  $\sigma : [0, T] \times \mathcal{B} \rightarrow \mathbb{H}$ .  $f$  and  $\sigma$  are appropriate functions and  $\mathcal{L}_b(\mathbb{K}, \mathbb{H})$  is the space of bounded functions from  $\mathbb{K}$  into  $\mathbb{H}$ . The initial data  $\{\varphi(t) : t \in (-\infty, 0]\}$  is an  $\mathcal{F}_0$ -adapted,  $\mathcal{B}$ -valued random variable independent of the Wiener process  $\omega$  with finite second moment and  $x_0$  is an  $\mathcal{F}_0$ -adapted,  $\mathbb{H}$  valued random variable independent of  $\omega$ .

Our objective in this work is to investigate the existence of mild solutions and optimal control for system (1) by using the Krasnoselskii-Shaefer fixed point theorem combined with the resolvent operator theory. Furthermore, to the best of our knowledge, the optimal controls for stochastic partial functional integrodifferential equations (1) with infinite delay are untreated in the literature, and this fact motivates us to extend the existing ones and make new development of the present work on this issue.

The remainder of this work is structured accordingly. We introduce some basic notations and required preliminaries in Section 2. In Section 3, we prove the existence of mild solutions for system (1). Section 4 displays the outcomes for optimal pairs of system governed by stochastic control system (1). Finally, an example is given in Section 5 to illustrate the obtained results.

## 2. PRELIMINARIES

Let  $\mathbb{H}$ ,  $\mathbb{K}$  be two real separable Hilbert spaces and we denote by  $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ ,  $\langle \cdot, \cdot \rangle_{\mathbb{K}}$  their inner products and by  $\|\cdot\|_{\mathbb{H}}$ ,  $\|\cdot\|_{\mathbb{K}}$  their corresponding vector norms respectively. Let  $L(\mathbb{K}, \mathbb{H})$  be the space of linear operators mapping  $\mathbb{K}$  into  $\mathbb{H}$  equipped with the usual norm  $\|\cdot\|_{\mathbb{H}}$ , and  $\mathcal{L}_b(\mathbb{H})$  denotes the Hilbert space of bounded linear operators from  $\mathbb{H}$  to  $\mathbb{H}$ .

Let  $\{w(t) : t \geq 0\}$  denote an  $\mathbb{K}$ -valued Wiener process defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with covariance operator  $Q$ , that is  $\mathbb{E} \langle w(t), x \rangle_{\mathbb{K}} \langle w(t), y \rangle_{\mathbb{K}} = t \wedge s \langle Qx, y \rangle_{\mathbb{H}}$ , for all  $x, y \in \mathbb{H}$ , where  $Q$  is a positive, self-adjoint, trace class operator on  $\mathbb{K}$ . In particular, we denote  $w(t)$  an  $\mathbb{K}$ -valued  $Q$ -Wiener process with respect to  $\{\mathcal{F}_t\}_{t \geq 0}$ .

In order to define stochastic integrals with respect to the  $Q$ -Wiener process  $w(t)$ , we introduce the subspace  $\mathbb{K}_0 = Q^{\frac{1}{2}}(\mathbb{K})$  of  $\mathbb{K}$  which is endowed with the inner product  $\langle \tilde{u}, \tilde{v} \rangle_{\mathbb{K}_0} = \langle Q^{-\frac{1}{2}}\tilde{u}, Q^{-\frac{1}{2}}\tilde{v} \rangle_{\mathbb{K}}$  is a Hilbert space. We assume that there exists a complete orthonormal system  $\{e_n\}_{n \geq 1}$  in  $\mathbb{K}$ , a bounded sequence of non-negative real numbers  $\{\lambda_n\}_{n \geq 1}$  such that  $Qe_n = \lambda_n e_n$  and a sequence  $\beta_n$  of independent Brownian motions such that

$$\langle w(t), e \rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle e_n, e \rangle \beta_n(t), \quad e \in \mathbb{K}, t \in J,$$

and  $\mathcal{F}_t = \mathcal{F}_t^w$ , where  $\mathcal{F}_t^w$  is the  $\sigma$ -algebra generated by  $\{w(s) : 0 \leq s \leq t\}$ . Let  $L_2^0 = L_2(\mathbb{K}_0, \mathbb{H})$  be the space of all Hilbert-Schmidt operators from  $\mathbb{K}_0$  to  $\mathbb{H}$  with the norm  $\|\psi\|_{L_2^0}^2 = \text{Tr}((\psi Q^{\frac{1}{2}})(\psi Q^{\frac{1}{2}})^*)$  for any  $\psi \in L_2^0$ . Obviously, for any bounded operator  $\psi \in \mathcal{L}_b(\mathbb{K}, \mathbb{H})$  this norm reduces to  $\|\psi\|_{L_2^0}^2 = \text{Tr}(\psi Q \psi^*)$ . Let  $L^p(\mathcal{F}_T, \mathbb{H})$  be the Banach space of all  $\mathcal{F}_T$ -measurable pth power integral random variables with values in the Hilbert space  $\mathbb{H}$ . Let  $C([0, T]; L^p(\mathcal{F}, \mathbb{H}))$  be the Banach space of continuous maps from  $[0, T]$  into  $L^p(\mathcal{F}, \mathbb{H})$  satisfying the condition  $\sup_{t \in J} \mathbb{E} \|x(t)\|_{\mathbb{H}}^p < \infty$ . In particular, we introduce the space  $\mathcal{C}(J, \mathbb{H})$  denoting the closed subspace of  $C([0, T]; L^p(\mathcal{F}, \mathbb{H}))$  consisting of measurable and  $\mathcal{F}_T$ -adapted  $\mathbb{H}$ -valued stochastic processes  $x \in C([0, T]; L^p(\mathcal{F}, \mathbb{H}))$  endowed with the norm  $\|x\|_{\mathcal{C}} = (\sup_{0 \leq t \leq T} \mathbb{E} \|x(t)\|_{\mathbb{H}}^p)^{\frac{1}{p}}$ . Then  $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$  is a Banach space. The notation  $B_r(x, \mathbb{H})$  stands for the closed ball with center  $x$  and radius  $r > 0$  in  $\mathbb{H}$ .

In the following, let  $\mathbb{Y}$  be a separable reflexive Hilbert space from which the controls  $u$  take values. The operator  $C \in L_{\infty}(J, L(\mathbb{Y}, \mathbb{H}))$ ,  $\|C\|_{\infty}$  stands for the norm of operator  $C$  on Banach space  $L_{\infty}(J, L(\mathbb{Y}, \mathbb{H}))$ , where  $L_{\infty}(J, L(\mathbb{Y}, \mathbb{H}))$  denotes the space of operator valued functions which are measurable in the strong operator topology and uniformly bounded on the interval  $J$ . Let  $L_{\mathcal{F}}^p(J, \mathbb{Y})$  be the closed subspace of  $L_{\mathcal{F}}^p(J \times \Omega, \mathbb{Y})$ , consisting of all measurable and  $\mathcal{F}_t$ -adapted,  $\mathbb{Y}$ -valued stochastic processes satisfying the condition  $\mathbb{E} \int_0^T \|u(t)\|_{\mathbb{Y}}^p dt < \infty$ , and endowed with the norm  $\|u\|_{L_{\mathcal{F}}^p(J, \mathbb{Y})} = (\mathbb{E} \int_0^T \|u(t)\|_{\mathbb{Y}}^p dt)^{\frac{1}{p}}$ . Let  $\mathbb{U}$  be a non-empty closed bounded convex subset of  $\mathbb{Y}$ . Now, we define the admissible control set as follows:

$$\mathbb{U}_{ad} = \{v(\cdot) \in L_{\mathcal{F}}^p(J, \mathbb{Y}); v(t) \in \mathbb{U} \text{ a.e. } t \in J\}.$$

Then  $Cu \in L^p(J, \mathbb{H})$  for all  $u \in \mathbb{U}_{ad}$ .

In this work, we will use an axiomatic definition for the  $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$  phase space, which is a seminormed linear space of  $\mathcal{F}_0$ -measurable functions mapping  $(-\infty, 0]$  into  $\mathbb{H}$  and satisfying the understanding fundamental axioms due to Hale and Kato (see e.g. in [7]).

**(A):** If  $x : (-\infty, \tilde{\sigma} + T] \rightarrow \mathbb{H}$ ,  $T > 0$  such that  $x|_{[\tilde{\sigma}, \tilde{\sigma} + T]} \in C([\tilde{\sigma}, \tilde{\sigma} + T], \mathbb{H})$  and  $x_{\tilde{\sigma}} \in \mathcal{B}$ , then for every  $t \in [\tilde{\sigma}, \tilde{\sigma} + T]$  the following conditions hold:

**(1):**  $x_t$  in  $\mathcal{B}$ ;

**(2):**  $\|x(t)\|_{\mathbb{H}} \leq \tilde{H}\|x_t\|_{\mathcal{B}}$ ;

**(3):**  $\|x_t\|_{\mathcal{B}} \leq K(t - \tilde{\sigma}) \sup\{\|x(s)\|_{\mathbb{H}} : \tilde{\sigma} \leq s \leq t\} + M(t - \tilde{\sigma})\|x_{\tilde{\sigma}}\|_{\mathcal{B}}$ , where  $\tilde{H} \geq 0$  is a constant;  $K, M : [0, \infty) \rightarrow [1, \infty)$ ,  $K$  is continuous and  $M$  is locally bounded;  $\tilde{H}, K, M$  are independent of  $x(\cdot)$ .

**(B):** For the function  $x(\cdot)$  in (A), the function  $t \rightarrow x_t$  is continuous from  $[\tilde{\sigma}, \tilde{\sigma} + T]$  into  $\mathcal{B}$ .

**(C):** The space  $\mathcal{B}$  is complete.

Next, to be able to prove the existence of the mild solutions for (1), we need to introduce partial integrodifferential equations and resolvent operators that will be used to develop the main results of this work.

Let  $X$  be Banach space. We denote by  $\mathcal{L}(X, \mathbb{Y})$  the Banach space of bounded linear operators from  $X$  into  $\mathbb{Y}$  endowed with operator norm and we abbreviate this notation to  $\mathcal{L}(X)$  when  $X = \mathbb{Y}$ .

In what follows,  $A$  and  $B(t)$  are closed linear operators on  $X$ .  $\mathbb{Y}$  represents the Banach space  $D(A)$  equipped with the graph norm defined by

$$\|y\|_{\mathbb{Y}} := \|Ay\|_X + \|y\|_X \quad \text{for } y \in \mathbb{Y}.$$

The notations  $\mathcal{C}([0, +\infty); \mathbb{Y})$ ,  $\mathcal{L}(\mathbb{Y}, X)$  stand for the space of all continuous functions from  $[0, +\infty)$  into  $\mathbb{Y}$ , the set of all bounded linear operators from  $\mathbb{Y}$  into  $X$ , respectively.

Assume that

**(R<sub>1</sub>)**  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  in  $X$ .

**(R<sub>2</sub>)** For all  $t \geq 0$ ,  $B(t)$  is a closed linear operator from  $D(A)$  to  $X$  and  $B(t) \in \mathcal{L}(\mathbb{Y}, X)$ . For any  $y \in \mathbb{Y}$ , the map  $t \rightarrow B(t)y$  is bounded uniformly continuous, differentiable and the derivative  $t \rightarrow B'(t)y$  is bounded uniformly continuous on  $\mathbb{R}^+$ .

We consider the following integrodifferential abstract Cauchy problem:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds & \text{for } t \geq 0 \\ x(0) = x_0 \in X. \end{cases} \quad (2)$$

**Definition 2.1.** [24] We call resolvent operator for the system (2), a bounded linear operator valued function

$$R : [0, +\infty) \rightarrow \mathcal{L}(X)$$

satisfying the following properties:

- (i)  $R(0) = I$  (identity operator on  $X$ ) and  $\|R(t)\| \leq Ne^{\beta t}$  for some constants  $N > 0$  and  $\beta$ .
- (ii) For each  $x \in X$ ,  $R(t)x$  is strongly continuous for  $t \geq 0$ .
- (iii)  $R(t) \in \mathcal{L}(Y)$  for  $t \geq 0$ . For  $x \in Y$ ,  $R(\cdot)x \in C^1([0, +\infty), X) \cap C([0, +\infty), Y)$  and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)x ds \\ &= R(t)Ax + \int_0^t R(t-s)K(s)x ds \quad \text{for } t \geq 0. \end{aligned}$$

**Theorem 2.1.** [9] Assume that  $(R_1)$  and  $(R_2)$  are satisfied. Then, there exists a unique resolvent operator for Eq. (2).

Now, we give the definition of mild solution for Eq. (1).

**Definition 2.2.** An  $\mathcal{F}_t$ -adapted stochastic process  $x : (-\infty, T] \rightarrow \mathbb{H}$  is called a mild solution of the system (1) with respect to  $u$  on  $(-\infty, T]$ , if  $x_0 = \varphi \in \mathcal{B}$ ,  $x|_J \in \mathcal{C}(J, \mathbb{H})$  for every  $u \in \mathbb{U}_{ad}$  there exists a  $T = T(u) > 0$  and

- (i):  $x(t)$  is measurable and adapted to  $\mathcal{F}_t$ ,  $t \geq 0$ .
- (ii):  $x(t) \in \mathbb{H}$  has càdlàg paths on  $t \in J$  a.s. and for each  $t \in J$ ,  $x(t)$  satisfies

$$\begin{aligned} x(t) &= R(t)\varphi(0) + \int_0^t R(t-s)C(s)u(s)ds \\ &\quad + \int_0^t R(t-s)\sigma(s, x_s)ds \\ &\quad + \int_0^t R(t-s)f(s, x_s)dw(s), \quad t \in J. \end{aligned} \tag{3}$$

The following result is a consequence of the phase space axioms.

**Lemma 2.2.** [23] Let  $x : (-\infty, T] \rightarrow \mathbb{H}$  be an  $\mathcal{F}_t$ -adapted measurable process such that the  $\mathcal{F}_0$ -adapted process  $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$  and  $x|_J \in \mathcal{C}(J, H)$ , then

$$\|x_s\|_{\mathcal{B}} \leq M_T \mathbb{E}\|\varphi\|_{\mathcal{B}} + \sup_{0 \leq s \leq T} K_T \mathbb{E}\|x(s)\|_{\mathbb{H}},$$

where  $M_T = \sup_{t \in J} M(t)$  and  $K_T = \sup_{t \in J} K(t)$ .

The following fundamental lemma plays an important role in the existence of mild solutions and the existence of stochastic optimal controls.

**Lemma 2.3.** [19] For any  $p \geq 1$  and for arbitrary  $L_2^0(\mathbb{K}, \mathbb{H})$ -valued predictable process  $\phi(\cdot)$  such that

$$\sup_{s \in [0, t]} \mathbb{E} \left\| \int_0^s \phi(v)dw(v) \right\|_{\mathbb{H}}^{2p} \leq (p(2p-1))^p \left( \int_0^t (\mathbb{E}\|\phi(s)\|_{L_2^0}^{2p})^{1/p} ds \right)^p, \quad t \geq 0.$$

In the rest of this paper, we denote by  $C_p = (p(2p-1))^p$ .

**Lemma 2.4** (Bochner's Theorem [16]). A measurable function  $V : J \rightarrow \mathbb{H}$  is Bochner integrable, if  $\|V\|_{\mathbb{H}}$  is Lebesgue integrable.

Next, we present the Krasnoleiskii-Schaefer-type fixed point theorem appeared in [4].

**Lemma 2.5** (Krasnoleiskii-Schaefer-type fixed point theorem [4]). *Let  $\phi_1, \phi_2$  be two operators such that*

- (i):  $\phi_1$  is a contraction, and
- (ii):  $\phi_2$  is completely continuous. Then either
  - (\*) : the operator equation  $\phi_1 x + \phi_2 x = x$  has a solution, or
  - (\*\*) : the set  $G = \{x \in \mathbb{H} : \alpha \phi_1(\frac{x}{\alpha}) + \alpha \phi_2 x\}$  is unbounded for  $\alpha \in (0, 1)$ .

### 3. EXISTENCE OF MILD SOLUTIONS FOR STOCHASTIC INTEGRODIFFERENTIAL SYSTEM

In this section, we prove the existence outcomes for stochastic system (1). Let us introduce the subsequent hypotheses.

- (A<sub>1</sub>): The resolvent operator is compact and there exists a constant  $M \geq 1$  such that  $\|R(t)\| \leq M$ .
- (A<sub>2</sub>): The function  $\sigma(t, \cdot) : \mathcal{B} \rightarrow \mathbb{H}$  is continuous for each  $t \in J$ , and for every  $\phi \in \mathcal{B}$ , the function  $t \rightarrow \sigma(t, \phi)$  is strongly measurable.
- (A<sub>3</sub>): There exist a function  $m_\sigma(\cdot) \in L^1(J, \mathbb{R}_+)$  and continuous nondecreasing function  $\Theta_\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$  such that

$$\mathbb{E} \|\sigma(t, \psi)\|_{\mathbb{H}}^p \leq m_\sigma(t) \Theta_\sigma(\|\psi\|_{\mathcal{B}}^p)$$

with

$$\int_1^\infty \frac{1}{\Theta_\sigma(s)} ds = \infty.$$

- (A<sub>4</sub>): The function  $\sigma : J \times \mathcal{B} \rightarrow \mathbb{H}$  is compact.
- (A<sub>5</sub>): The function  $f : J \times \mathcal{B} \rightarrow \mathcal{L}_b(\mathbb{K}, \mathbb{H})$  is continuous and there exists a constant  $L_f > 0$  such that

$$\mathbb{E} \|f(t, \phi_1) - f(t, \phi_2)\|_{\mathbb{H}}^p \leq L_f \mathbb{E} \|\phi_1 - \phi_2\|_{\mathcal{B}}^p, \quad t \in J, \quad \phi_1, \phi_2 \in \mathcal{B}.$$

**Theorem 3.1.** *Assume that assumptions (R<sub>1</sub>), (R<sub>2</sub>) and (A<sub>1</sub>) – (A<sub>5</sub>) hold. Then, for each  $u \in \mathbb{U}_{ad}$ , Eq. (1) has at least one mild solution on  $J$  with respect to  $u$ , provided that*

$$16^{p-1} C_p K_T^p M^p T^{p/2} L_f < 1.$$

*Proof.* Consider the space  $\mathcal{BC} = \{x(t) : (-\infty, T] \rightarrow \mathbb{H}; x(0) = \varphi(0), x(t)|_J \in C(J, \mathbb{H})\}$  endowed with the uniform convergence topology and define the operator  $\Psi : \mathcal{BC} \rightarrow \mathcal{BC}$  by

$$(\Psi x)(t) = \begin{cases} \varphi(t), & t \in [-\infty, 0], \\ R(t)\varphi(0) + \int_0^t R(t-s)C(s)u(s)ds \\ + \int_0^t R(t-s)\sigma(s, \bar{x}_s)ds + \int_0^t R(t-s)f(s, \bar{x}_s)dw(s), & t \in J, \end{cases}$$

where  $\bar{x}(t) : (-\infty, T] \rightarrow \mathbb{H}$  is given such that  $\bar{x}(0) = \varphi$  and  $\bar{x} = x$  on  $J$ . Using Hölder's inequality, we have

$$\begin{aligned} \mathbb{E} \left\| \int_0^t R(t-s)C(s)u(s)ds \right\|_{\mathbb{H}}^p &\leq M^p T^{p-1} \|C\|_{\infty}^p \mathbb{E} \int_0^t \|u(s)\|_{\mathbb{Y}}^p ds \\ &\leq M^p T^{p-1} \|C\|_{\infty}^p \|u\|_{L_{\mathcal{F}}^p(J, \mathbb{Y})}^p, \end{aligned}$$

where  $\|C\|_\infty$  is the norm of operator in Banach space  $L_\infty(J, L(\mathbb{Y}, \mathbb{H}))$ .

From Lemma 2.4, it follows that  $R(t-s)C(s)u(s)$  is Bochner integrable with respect to  $s \in [0, t]$  for all  $t \in J$ . Hence, we conclude that  $\Psi$  is defined from  $\mathcal{BC}$  into  $\mathcal{BC}$ .

Let  $\tilde{\varphi} : (-\infty, 0) \rightarrow \mathbb{H}$  be the extension of  $\varphi$  to  $(-\infty, 0]$  such that  $\tilde{\varphi}(0) = \varphi(0)$  on  $J$ . Now, we decompose  $\Psi$  as  $\Psi_1$  and  $\Psi_2$  i.e.  $\Psi = \Psi_1 + \Psi_2$  where

$$\begin{aligned} (\Psi_1 x)(t) &= R(t)\varphi(0) + \int_0^t R(t-s)f(s, \bar{x}_s)dw(s), & t \in J, \\ (\Psi_2 x)(t) &= \int_0^t R(t-s)C(s)u(s)ds + \int_0^t R(t-s)\sigma(s, \bar{x}_s)ds, & t \in J. \end{aligned}$$

In order to apply Lemma 2.5, we will verify that  $\Psi_1$  is a contraction while  $\Psi_2$  is a completely continuous operator. The proof is splitted into six steps.

**Step 1.**  $\Psi_1$  is a contraction on  $\mathcal{BC}$ .

Let  $t \in [0, T]$  and  $v^1, v^2 \in \mathcal{BC}$ . From  $(A_5)$  and Lemma 2.2, we have

$$\begin{aligned} & \mathbb{E}\|(\Psi_1 v^1)(t) - (\Psi_1 v^2)(t)\|_{\mathbb{H}}^p \\ &= \mathbb{E}\left\|\int_0^t R(t-s)[f(s, \bar{v}_s^1) - f(s, \bar{v}_s^2)]dw(s)\right\|_{\mathbb{H}}^p \\ &\leq C_p \left[ \int_0^t \left[ \|R(t-s)\|_{\mathbb{H}}^p \mathbb{E}\|f(s, \bar{v}_s^1) - f(s, \bar{v}_s^2)\|_{\mathbb{H}}^p \right]^{2/p} ds \right]^{p/2} \\ &\leq C_p M^p \left[ \int_0^t \left[ \mathbb{E}\|f(s, \bar{v}_s^1) - f(s, \bar{v}_s^2)\|_{\mathbb{H}}^p \right]^{2/p} ds \right]^{p/2} \\ &\leq C_p M^p T^{p/2-1} L_f \int_0^t \|\bar{v}_s^1 - \bar{v}_s^2\|_{\mathcal{B}}^p ds \\ &\leq 2^{p-1} C_p M^p T^{p/2} L_f \sup_{s \in [0, T]} \mathbb{E}\|\bar{v}^1(s) - \bar{v}^2(s)\|_{\mathbb{H}}^p \\ &= 2^{p-1} C_p M^p T^{p/2} L_f \sup_{s \in [0, T]} \mathbb{E}\|v^1(s) - v^2(s)\|_{\mathbb{H}}^p \\ &\quad \text{since } \bar{v}(s) = v(s) \text{ on } J \\ &= 2^{p-1} C_p M^p T^{p/2} L_f \|v^1 - v^2\|_{\mathcal{C}}^p. \end{aligned}$$

Taking the supremum over  $t$ , we get that

$$\|\Psi_1 v^1 - \Psi_1 v^2\|_{\mathcal{C}}^p \leq \theta_0 \|v^1 - v^2\|_{\mathcal{C}}^p,$$

where  $\theta_0 = 2^{p-1} C_p M^p T^{p/2} L_f < 1$ . Therefore,  $\Psi_1$  is a contraction on  $\mathcal{BC}$ .

**Step 2.**  $\Psi_2$  maps bounded sets into bounded sets in  $\mathcal{BC}$ .

Indeed, it is enough to show that there exists a positive constant  $\kappa$  such that for each  $x \in B_r(0, \mathcal{BC})$  one has  $\|\Psi_2 x\|_{\mathcal{C}}^p \leq \kappa$ . If  $x \in B_r(0, \mathcal{BC})$ , by Lemma 2.2, it follows that

$$\|\bar{x}_s\|_{\mathcal{B}}^p \leq 2^{p-1} (M_T \|\varphi\|_{\mathcal{B}})^p + 2^{p-1} K_T^p r := r_1.$$



By  $(\mathbf{A}_1) - (\mathbf{A}_5)$ , we have for  $t \in J$

$$\begin{aligned}
 \mathbb{E} \|(\Psi_2 x)(t)\|_{\mathbb{H}}^p &\leq 2^{p-1} \mathbb{E} \left\| \int_0^t R(t-s) C(s) u(s) ds \right\|_{\mathbb{H}}^p \\
 &\quad + 2^{p-1} \mathbb{E} \left\| \int_0^t R(t-s) \sigma(s, \bar{x}_s) ds \right\|_{\mathbb{H}}^p \\
 &\leq 2^{p-1} M^p T^{p-1} \mathbb{E} \int_0^t \|C(s) u(s)\|_{\mathbb{H}}^p ds \\
 &\quad + 2^{p-1} M^p T^{p-1} \int_0^t \mathbb{E} \|\sigma(s, \bar{x}_s)\|_{\mathbb{H}}^p ds \\
 &\leq 2^{p-1} M^p T^{p-1} \|C\|_{\infty}^p \mathbb{E} \int_0^t \|u(s)\|_{\mathbb{Y}}^p ds \\
 &\quad + 2^{p-1} M^p T^{p-1} \int_0^t m_{\sigma}(s) \Theta_{\sigma}(\|\bar{x}_s\|_{\mathcal{B}}^p) ds \\
 &\leq 2^{p-1} M^p T^{p-1} \|C\|_{\infty}^p \|u\|_{L_{\mathcal{F}}^p(J, \mathbb{Y})}^p \\
 &\quad + 2^{p-1} M^p T^{p-1} \Theta_{\sigma}(r_1) \int_0^t m_{\sigma}(s) ds := \kappa.
 \end{aligned}$$

Hence, for each  $x \in B_r(0, \mathcal{BC})$ , we have  $\|\Psi_2 x\|_{\mathcal{C}}^p \leq \kappa$ .

**Step 3.**  $\Psi_2$  maps  $B_r(0, \mathcal{BC})$  into a relatively compact set in  $\mathbb{H}$ .

It follows from the strong continuity of  $(R(t))_{t \geq 0}$  and condition  $(\mathbf{A}_4)$  that, the set  $\{R(t-s)\sigma(s, \phi); t, s \in [0, T], \|\psi\|_{\mathcal{B}}^p \leq r^*\}$  is relatively compact in  $\mathbb{H}$ . Moreover for  $x \in B_r(0, \mathcal{BC})$ , from the mean value theorem for the Bochner integral, we can infer that

$$(\Psi_2 x)(t) \in \overline{\text{co}}\{R(t-s)\sigma(s, \phi); t, s \in [0, T], \|\psi\|_{\mathcal{B}}^p \leq r^*\}$$

for all  $t \in J$ ;  $\overline{\text{co}}$  denotes the convex hull. As a result, we conclude that  $\{(\Psi_2 x)(t) : x \in B_r(0, \mathcal{BC})\}$  is the relatively compact set in  $\mathbb{H}$  for every  $t \in J$ .

**Step 4.**  $\Psi_2$  maps bounded sets into equicontinuous sets of  $\mathcal{BC}$ .

Let  $\epsilon$  be a positive number such that  $0 < \epsilon < t < T$ . From step 3,  $(\Psi_2 B_r(0, \mathcal{BC}))(t)$  is relatively compact for each  $t$  and by the strong continuity of  $R(t)$ , we can choose  $0 < \delta < T - t$  with

$$\|R(t+l)x - R(t)x\|_{\mathbb{H}} \leq \epsilon$$

for  $x \in (\Psi_2 B_r(0, \mathcal{BC}))(t)$  when  $0 < l < \delta$ .

For any  $x \in B_r(0, \mathcal{BC})$ . Using  $(\mathbf{A}_1) - (\mathbf{A}_5)$ , we obtain

$$\begin{aligned}
 & \mathbb{E} \|(\Psi_2 x)(t+l) - (\Psi_2 x)(t)\|_{\mathbb{H}}^p \\
 & \leq 6^{p-1} \mathbb{E} \left\| \int_0^{t-\epsilon} [R(t+l-s) - R(t-s)] C(s) u(s) ds \right\|_{\mathbb{H}}^p \\
 & \quad + 6^{p-1} \mathbb{E} \left\| \int_{t-\epsilon}^t [R(t+l-s) - R(t-s)] C(s) u(s) ds \right\|_{\mathbb{H}}^p \\
 & \quad + 6^{p-1} \mathbb{E} \left\| \int_t^{t+l} R(t+l-s) C(s) u(s) ds \right\|_{\mathbb{H}}^p \\
 & \quad + 6^{p-1} \mathbb{E} \left\| \int_0^{t-\epsilon} [R(t+l-s) - R(t-s)] \sigma(s, \bar{x}_s) ds \right\|_{\mathbb{H}}^p \\
 & \quad + 6^{p-1} \mathbb{E} \left\| \int_{t-\epsilon}^t [R(t+l-s) - R(t-s)] \sigma(s, \bar{x}_s) ds \right\|_{\mathbb{H}}^p \\
 & \quad + 6^{p-1} \mathbb{E} \left\| \int_t^{t+l} R(t+l-s) \sigma(s, \bar{x}_s) ds \right\|_{\mathbb{H}}^p \\
 & := P_1 + P_2 + P_3 + P_4 + P_5 + P_6.
 \end{aligned} \tag{4}$$

By Hölder's inequality, we have the following estimates

$$\begin{aligned}
 P_1 & \leq 6^{p-1} \|C\|_{\infty}^p \left( \int_0^{t-\epsilon} \|R(t+l-s) - R(t-s)\|_{\mathbb{H}}^{\frac{p}{p-1}} ds \right)^{p-1} \times \mathbb{E} \int_0^{t-\epsilon} \|u(s)\|_{\mathbb{Y}}^p ds \\
 & \leq 6^{p-1} \|C\|_{\infty}^p \left( \int_0^{t-\epsilon} \|R(t+l-s) - R(t-s)\|_{\mathbb{H}}^{\frac{p}{p-1}} ds \right)^{p-1} \times \mathbb{E} \int_0^{t-\epsilon} \|u(s)\|_{\mathbb{Y}}^p ds \\
 & \leq 6^{p-1} \|C\|_{\infty}^p \epsilon^p (t-\epsilon)^{p-1} \mathbb{E} \int_0^{t-\epsilon} \|u(s)\|_{\mathbb{Y}}^p ds,
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 P_2 & \leq 6^{p-1} \|C\|_{\infty}^p \epsilon^{p-1} \mathbb{E} \int_{t-\epsilon}^t \|R(t+l-s) - R(t-s)\|_{\mathbb{H}}^p \|u(s)\|_{\mathbb{Y}}^p ds \\
 & \leq 12^{p-1} M^p \epsilon^{p-1} \mathbb{E} \int_{t-\epsilon}^t \|u(s)\|_{\mathbb{Y}}^p ds
 \end{aligned} \tag{6}$$

and

$$\begin{aligned}
 P_3 & \leq 6^{p-1} \|C\|_{\infty}^p l^{p-1} \mathbb{E} \int_t^{t+l} \|R(t+l-s)\|_{\mathbb{H}}^p \|u(s)\|_{\mathbb{Y}}^p ds \\
 & \leq 6^{p-1} \|C\|_{\infty}^p l^{p-1} M^p \mathbb{E} \int_t^{t+l} \|u(s)\|_{\mathbb{Y}}^p ds.
 \end{aligned} \tag{7}$$

On the other hand, we obtain from Hölder inequality in view of assumptions  $(\mathbf{A}_3)$  and  $(\mathbf{A}_4)$

$$\begin{aligned}
 P_4 & \leq 6^{p-1} (t-\epsilon)^{p-1} \int_0^{t-\epsilon} \mathbb{E} \| [R(t+l-s) - R(t-s)] \sigma(s, \bar{x}_s) \|_{\mathbb{H}}^p ds \\
 & \leq 6^{p-1} (t-\epsilon)^{p-1} \int_0^{t-\epsilon} \mathbb{E} \| [R(t+l-s) - R(t-s)] \sigma(s, \bar{x}_s) \|_{\mathbb{H}}^p ds \\
 & \leq 6^{p-1} \epsilon^p (t-\epsilon)^{p-1}.
 \end{aligned} \tag{8}$$

According to  $(A_1)$  and  $(A_5)$  and the Hölder's inequality, we have

$$\begin{aligned} P_5 &\leq 6^{p-1} \epsilon^{p-1} \int_{t-\epsilon}^t \|R(t+l-s) - R(t-s)\|_{\mathbb{H}}^p \mathbb{E} \|\sigma(s, \bar{x}_s)\|_{\mathbb{H}}^p ds \\ &\leq 12^{p-1} M^p \epsilon^{p-1} \int_{t-\epsilon}^t m_\sigma(s) \Theta_\sigma(\|\bar{x}_s\|_{\mathcal{B}}^p) ds \\ &\leq 12^{p-1} M^p \epsilon^{p-1} \Theta_\sigma(r_1) \int_{t-\epsilon}^t m_\sigma(s) ds, \end{aligned} \quad (9)$$

and

$$\begin{aligned} P_6 &\leq 6^{p-1} l^{p-1} \int_t^{t+l} \|R(t+l-s)\|_{\mathbb{H}}^p \mathbb{E} \|\sigma(s, \bar{x}_s)\|_{\mathbb{H}}^p ds \\ &\leq 6^{p-1} l^{p-1} M^p \int_t^{t+l} m_\sigma(s) \Theta_\sigma(\|\bar{x}_s\|_{\mathcal{B}}^p) ds \\ &\leq 6^{p-1} l^{p-1} M^p \Theta_\sigma(r_1) \int_t^{t+l} m_\sigma(s) ds. \end{aligned} \quad (10)$$

Gathering the above inequalities (5)-(10), we obtain

$$\begin{aligned} \mathbb{E} \|(\Psi_2 x)(t+l) - (\Psi_2 x)(t)\|_{\mathbb{H}}^p &\leq 6^{p-1} \epsilon^p (t-\epsilon)^{p-1} \\ &\quad + 12^{p-1} M^p \epsilon^{p-1} \mathbb{E} \int_{t-\epsilon}^t \|u(s)\|_{\mathbb{Y}}^p ds \\ &\quad + 6^{p-1} \|C\|_{\infty}^p l^{p-1} M^p \mathbb{E} \int_t^{t+l} \|u(s)\|_{\mathbb{Y}}^p ds \\ &\quad + 6^{p-1} \|C\|_{\infty}^p \epsilon^p (t-\epsilon)^{p-1} \mathbb{E} \int_0^{t-\epsilon} \|u(s)\|_{\mathbb{Y}}^p ds \\ &\quad + 12^{p-1} M^p \epsilon^{p-1} \Theta_\sigma(r_1) \int_{t-\epsilon}^t m_\sigma(s) ds \\ &\quad + 6^{p-1} l^{p-1} M^p \Theta_\sigma(r_1) \int_t^{t+l} m_\sigma(s) ds. \end{aligned} \quad (11)$$

Consequently, the right-hand side of (11) is independent of  $x \in B_r(0, \mathcal{BC})$  and tends to zero as  $l \rightarrow 0$  and  $\epsilon$  sufficiently small. Thus, the set  $\{\Psi_2 x : x \in B_r(0, \mathcal{BC})\}$  is equicontinuous.

**Step 5.**  $\Psi_2 : \mathcal{BC} \rightarrow \mathcal{BC}$ .

Let  $\{x^{(n)}\} \subseteq B_r(0, \mathcal{BC})$  with  $x^{(n)} \rightarrow x$  as  $n \rightarrow \infty$  in  $\mathcal{BC}$ . From Axiom (A), it is easy to see that  $(x^{(n)})_s \rightarrow \bar{x}_s$  uniformly for  $s \in (-\infty, T]$  as  $n \rightarrow \infty$ . By the conditions  $(A_1)$  and  $(A_2)$ , we have

$$\sigma(s, \overline{x^{(n)}}_s) \rightarrow \sigma(s, \bar{x}_s) \text{ as } n \rightarrow \infty$$

for each  $s \in [0, t]$ , and since

$$\mathbb{E} \|\sigma(s, \overline{x^{(n)}}_s) - \sigma(s, \bar{x}_s(\eta, \cdot))\|_{\mathbb{H}}^p \leq 2^{p-1} \Theta_\sigma(r_1) m_\sigma(s), \quad s \in [0, T],$$

by the dominated convergence theorem we obtain

$$\begin{aligned} \mathbb{E} \|(\Psi_2 x^{(n)})(t) - (\Psi_2 x)(t)\|_{\mathbb{H}}^p &= \mathbb{E} \left\| \int_0^t R(t-s) [\sigma(s, \overline{x^{(n)}}_s) - \sigma(s, \bar{x}_s)] ds \right\|_{\mathbb{H}}^p \\ &\leq T^{p-1} \int_0^t \|R(t-s)\|_{\mathbb{H}}^p \mathbb{E} \|\sigma(s, \overline{x^{(n)}}_s) - \sigma(s, \bar{x}_s)\|_{\mathbb{H}}^p ds \\ &\leq M^p T^{p-1} \int_0^t \mathbb{E} \|\sigma(s, \overline{x^{(n)}}_s) - \sigma(s, \bar{x}_s)\|_{\mathbb{H}}^p ds. \end{aligned}$$

Therefore,

$$\|\Psi_2 x^{(n)} - \Psi_2 x\|_{\mathcal{C}}^p = \sup_{t \in J} \mathbb{E} \|(\Psi_2 x^{(n)})(t) - (\Psi_2 x)(t)\|_{\mathbb{H}}^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence,  $\Psi_2$  is continuous and  $\Psi_2$  is a completely continuous operator.

**Step 6.** Consider the following set

$$G = \{x \in \mathcal{BC} : \alpha \Psi_1\left(\frac{x}{\alpha}\right) + \alpha \Psi_2(x) = x, \text{ for some } \alpha \in (0, 1)\}.$$

We show that  $G$  is bounded in  $J$ .

We consider the following nonlinear operator equation

$$x(t) = \alpha \Psi x(t), \quad \alpha \in (0, 1), \quad (12)$$

where  $\Psi$  is already defined. Next, we give a priori estimate for the solution of the above equation. In fact, let  $x \in \mathcal{BC}$  be a possible solution of  $x = \alpha \Psi x$  for some  $\alpha \in (0, 1)$ . This implies that for each  $t \in J$ , we have

$$\begin{aligned} x(t) &= \alpha R(t) \varphi(0) + \alpha \int_0^t R(t-s) C(s) u(s) ds \\ &\quad + \alpha \int_0^t R(t-s) \sigma(s, x_s) ds + \alpha \int_0^t R(t-s) f(s, x_s) dw(s), \quad t \in J. \end{aligned} \quad (13)$$

By  $(\mathbf{A}_1) - (\mathbf{A}_5)$  and (13), we have for  $t \in J$ ,

$$\begin{aligned}
 \mathbb{E}\|x(t)\|_{\mathbb{H}}^p &\leq 4^{p-1}\mathbb{E}\|R(t)\varphi(0)\|_{\mathbb{H}}^p + 4^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)C(s)u(s)ds\right\|_{\mathbb{H}}^p \\
 &\quad + 4^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)\sigma(s, x_s)ds\right\|_{\mathbb{H}}^p \\
 &\quad + 4^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)f(s, x_s)dw(s)\right\|_{\mathbb{H}}^p \\
 &\leq 4^{p-1}M^p\mathbb{E}\|\varphi\|_{\mathbb{H}}^p + 4^{p-1}M^pT^{p-1}\|C\|_{\infty}^p\mathbb{E}\int_0^t \|u(s)\|_{\mathbb{Y}}^p ds \\
 &\quad + 4^{p-1}M^pT^{p-1}\int_0^t \mathbb{E}\|\sigma(s, \bar{x}_s)\|_{\mathbb{H}}^p ds \\
 &\quad + 8^{p-1}C_pM^p\left[\int_0^t \left[\mathbb{E}\|f(s, \bar{x}_s) - f(s, 0)\|_{\mathbb{H}}^p\right.\right. \\
 &\quad \left.\left.+ \mathbb{E}\|f(s, 0)\|_{\mathbb{H}}^p\right]^{2/p} ds\right]^{p/2} \\
 &\leq 4^{p-1}M^p(\tilde{H}\|\varphi\|_{\mathcal{B}})^p + 4^{p-1}M^pT^{p-1}\|C\|_{\infty}^p\|u\|_{L_{\mathcal{F}}^p(J, \mathbb{Y})}^p \\
 &\quad + 4^{p-1}M^pT^{p-1}\int_0^t m_{\sigma}(s)\Theta_{\sigma}(\|\bar{x}_s\|_{\mathcal{B}}^p)ds \\
 &\quad + 8^{p-1}C_pM^pT^{p/2-1}L_f\int_0^t \|\bar{x}_s\|_{\mathcal{B}}^p ds \\
 &\quad + 8^{p-1}C_pM^pT^{p/2-1}\int_0^t \mathbb{E}\|f(s, 0)\|_{\mathbb{H}}^p ds.
 \end{aligned}$$

We have by Lemma 2.2 that

$$\begin{aligned}
 \sup\{\|\bar{x}_s\|_{\mathcal{B}}^p : 0 \leq s \leq t\} &\leq 2^{p-1}(M_T\|\varphi\|_{\mathcal{B}})^p \\
 &\quad + 2^{p-1}K_T^p \sup\{\|x(s)\|_{\mathbb{H}}^p : 0 \leq s \leq t\}.
 \end{aligned}$$

Now, consider the function defined by

$$z(t) = 2^{p-1}(M_T\|\varphi\|_{\mathcal{B}})^p + 2^{p-1}K_T^p \sup\{\|x(s)\|_{\mathbb{H}}^p : 0 \leq s \leq t\}, \quad 0 \leq t \leq T.$$

For each  $t \in [0, T]$ , we have

$$\begin{aligned}
 z(t) &\leq 2^{p-1}(M_T\|\varphi\|_{\mathcal{B}})^p + 2^{p-1}K_T^p\theta \\
 &\quad + 16^{p-1}C_pM^pK_T^pT^{p/2}L_fz(t) \\
 &\quad + 8^{p-1}M^pK_T^pT^{p-1}\int_0^t m_{\sigma}(s)\Theta_{\sigma}(z(s))ds,
 \end{aligned}$$

where

$$\begin{aligned}
 \theta &= 4^{p-1}M^p(\tilde{H}\|\varphi\|_{\mathcal{B}})^p \\
 &\quad + 4^{p-1}M^pT^{p-1}\|C\|_{\infty}^p\|u\|_{L_{\mathcal{F}}^p(J, \mathbb{Y})}^p \\
 &\quad + 8^{p-1}C_pM^pT^{p/2-1}\int_0^t \mathbb{E}\|f(s, 0)\|_{\mathbb{H}}^p ds.
 \end{aligned}$$

Since  $\gamma = 16^{p-1}C_p M^p K_T^p T^{p/2} L_f < 1$ , we obtain

$$\begin{aligned} z(t) \leq & \frac{1}{1-\gamma} \left[ 2^{p-1} (M_T \|\varphi\|_{\mathcal{B}})^p + 2^{p-1} K_T^p \theta \right. \\ & \left. + 8^{p-1} M^p K_T^p T^{p-1} \int_0^t m_{\sigma}(s) \Theta_{\sigma}(z(s)) ds \right]. \end{aligned}$$

Detoting by  $\xi(t)$  the right-hand side of the above inequality, we have

$$z(t) \leq \xi(t), \text{ for all } t \in J,$$

and

$$\begin{aligned} \xi(0) &= \frac{1}{1-\gamma} \left[ 2^{p-1} (M_T \|\varphi\|_{\mathcal{B}})^p + 2^{p-1} K_T^p \theta \right] \\ \xi'(t) &\leq \frac{1}{1-\gamma} 8^{p-1} M^p K_T^p T^{p-1} m_{\sigma}(t) \Theta_{\sigma}(\xi(t)), \quad t \in J. \end{aligned}$$

Therefore, we have

$$\xi'(t) \leq \frac{1}{1-\gamma} 8^{p-1} M^p K_T^p T^{p-1} m_{\sigma}(t) \Theta_{\sigma}(\xi(t)), \text{ for each } t \in J.$$

This implies that

$$\int_{\xi(0)}^{\xi(t)} \frac{1}{\Theta_{\sigma}(\mu)} d\mu \leq \int_0^T \frac{1}{1-\gamma} 8^{p-1} M^p K_T^p T^{p-1} m_{\sigma}(t) dt < \infty.$$

Then, we deduce from the above inequality that there is a positive constant  $\Gamma$  independent of  $t$  such that  $\xi(t) \leq \Gamma$ ,  $t \in J$ . Hence, we have  $\|x\|_{\mathcal{C}}^p \leq z(t) \leq \xi(t) \leq \Gamma$ , where  $\Gamma$  depends only on  $M$ ,  $p$ ,  $C_p$ ,  $K_T$ ,  $T$  and on the function  $\Theta_{\sigma}(\cdot)$ . Thus, the set  $G$  is bounded on  $J$ . Finally, by Lemma 2.5, we deduce that  $\Psi$  has a fixed point  $x(\cdot) \in \mathcal{BC}$ , which is a mild solution of system (1). The proof is complete.  $\square$

#### 4. EXISTENCE OF STOCHASTIC OPTIMAL CONTROLS

In this section, we investigate the existence of stochastic optimal controls for the system (1). We consider the Lagrange problem  $(\mathcal{LP})$  associated to system (1):

$$(\mathcal{LP}) \begin{cases} \text{Find an optimal pair } (x^0, u^0) \in \mathcal{BC} \times \mathbb{U}_{ad} \text{ such that} \\ \mathcal{J}(x^0, u^0) \leq \mathcal{J}(x^u, u), \text{ for all } u \in \mathbb{U}_{ad}, \end{cases}$$

where

$$\mathcal{J}(x^u, u) = \mathbb{E} \int_0^T L(t, x_t^u, x^u(t), u(t)) dt,$$

is the cost function and  $x^u$  denotes the mild solution of system (1) corresponding to the control  $u \in \mathbb{U}_{ad}$ .

For the existence of solutions to problem  $(\mathcal{LP})$ , we make the following assumptions.

- (L1): The functional  $L : J \times \mathcal{B} \times \mathbb{H} \times \mathbb{Y} \rightarrow \mathbb{R} \cup \{\infty\}$  is Borel measurable.
- (L2):  $L(t, \cdot, \cdot, \cdot)$  is sequentially lower semicontinuous on  $\mathcal{B} \times \mathbb{H} \times \mathbb{Y}$  for almost all  $t \in J$ .
- (L3):  $L(t, x, y, \cdot)$  is convex on  $\mathbb{Y}$  for each  $x \in \mathcal{B}$ ,  $y \in \mathbb{H}$  and almost all  $t \in J$ .

**(L4):** There exist constants  $\beta_1, \beta_2 \geq 0$ ,  $\beta_3 > 0$  and  $\nu \in L^1(J, \mathbb{R})$  non-negative such that

$$L(t, x, y, u) \geq \nu(t) + \beta_1 \|x\|_{\mathcal{B}} + \beta_2 \|y\|_{\mathbb{H}} + \beta_3 \|u\|_{\mathbb{Y}}^p.$$

We have the following result on the existence of optimal controls for problem  $(\mathcal{LP})$ .

**Theorem 4.1.** Assume that **(L1)-(L4)** and the assumptions of Theorem 3.1 hold. Suppose that  $C$  be a compact operator. Then, the Lagrange problem  $(\mathcal{LP})$  admits at least one optimal pair on  $\mathcal{BC} \times \mathbb{U}_{ad}$ .

*Proof.* If  $\inf\{\mathcal{J}(x^u, u) | u \in \mathbb{U}_{ad}\} = +\infty$ , there is nothing to prove.

Without loss of generality, assume that  $\inf\{\mathcal{J}(x^u, u) | u \in \mathbb{U}_{ad}\} = \epsilon < +\infty$ .

By (L1)-(L4), we have

$$\begin{aligned} \mathcal{J}(x^u, u) &\geq \int_0^T \nu(t) dt + \beta_1 \int_0^T \|x_t^u(t)\|_{\mathcal{B}} dt + \beta_2 \int_0^T \|x^u(t)\|_{\mathbb{H}} dt \\ &\quad + \beta_3 \int_0^T \|u(t)\|_{\mathbb{Y}}^p dt \geq -\alpha > -\infty, \end{aligned}$$

where  $\alpha$  is a positive constant. Then,  $\epsilon \geq -\alpha > -\infty$ . Additionally, by using definition of infimum, there is a minimizing sequence of feasible pair  $\{(x^n, u^n)\} \subset \mathcal{S}_{ad}$ , such that

$$\mathcal{J}(x^n, u^n) \rightarrow \epsilon \text{ as } n \rightarrow +\infty,$$

where  $\mathcal{S}_{ad} = \{(x, u) : x \text{ is a mild solution of system (1) corresponding to } u \in \mathbb{U}_{ad}\}$ . Since  $\{u^n\}$  is bounded in  $L_{\mathcal{F}}^p(J, \mathbb{Y})$  for  $\{u^n\} \subseteq \mathbb{U}_{ad}$ , hence there exists a subsequence, relabeled as  $\{u^n\}$ , and  $u^0 \in L_{\mathcal{F}}^p(J, \mathbb{Y})$  such that

$$u^n \xrightarrow{w} u^0 \text{ in } L_{\mathcal{F}}^p(J, \mathbb{Y}) \text{ as } n \rightarrow \infty.$$

Since  $\mathbb{U}_{ad}$  is closed and convex, by Mazur Lemma, we conclude that  $u^0 \in \mathbb{U}_{ad}$ .

Now, we assume that  $x^n$  are the mild solutions of Eq. (1) corresponding to  $u^n$  and  $x^n$  satisfied the following equation:

$$\begin{aligned} x^n(t) &= R(t)\varphi(0) + \int_0^t R(t-s)C(s)u^n(s)ds \\ &\quad + \int_0^t R(t-s)\sigma(s, \overline{x_s^n})ds + \int_0^t R(t-s)f(s, \overline{x_s^n})dw(s), \quad t \in J. \end{aligned}$$

To simplify, we set  $\sigma_n(s) = \sigma(s, \overline{x_s^n})$ , by **(A4)**, we obtain that  $\sigma_n$  is a compact operator from  $J$  into  $\mathbb{H}$ . Therefore, by the compactness of  $\sigma_n$ , there exists a subsequence, relabeled as  $\{\sigma_n(\cdot)\}$ , and  $\overline{\sigma} \in \mathbb{H}$  such that

$$\sigma_n(\cdot) \rightarrow \overline{\sigma} \text{ in } \mathbb{H} \text{ as } n \rightarrow \infty.$$

Next, we consider the following controlled system

$$\begin{cases} x'(t) = \left[ Ax(t) + \int_0^t B(t-s)x(s)ds + C(t)u^0(t) \right. \\ \quad \left. + \overline{\sigma}(t) \right] dt + f(t, x_t)dw(t) \text{ for } t \in J, u^0 \in \mathbb{U}_{ad}, \\ x_0 = \varphi \in \mathcal{B}. \end{cases} \quad (14)$$

Then, by Theorem 3.1 the above system has a mild solution which is given by

$$\begin{aligned}\tilde{x}(t) &= R(t)\varphi(0) + \int_0^t R(t-s)C(s)u^0(s)ds \\ &\quad + \int_0^t R(t-s)\bar{\sigma}(s)ds + \int_0^t R(t-s)f(s, \tilde{x}_s)dw(s), \quad t \in J.\end{aligned}$$

Now, we show that  $x^n$  converges to  $\tilde{x}$  in  $\mathcal{BC}$  as  $n \rightarrow \infty$ . So, for each  $t \in J$ ,  $x^n(\cdot), \tilde{x}(\cdot) \in \mathcal{BC}$ , we have

$$\mathbb{E}\|x^n(t) - \tilde{x}\|_{\mathbb{H}}^p \leq \vartheta_n^1(t) + \vartheta_n^2(t) + \vartheta_n^3(t),$$

where

$$\begin{aligned}\vartheta_n^1(t) &= 3^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)C(s)[u^n(s) - u^0(s)]ds\right\|_{\mathbb{H}}^p, \\ \vartheta_n^2(t) &= 3^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)[\sigma_n(s) - \bar{\sigma}(s)]ds\right\|_{\mathbb{H}}^p, \\ \vartheta_n^3(t) &= 3^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)[f(s, \tilde{x}_s^n) - f(s, \tilde{x}_s)]dw(s)\right\|_{\mathbb{H}}^p.\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}\vartheta_n^1(t) &= 3^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)B(s)[u^n(s) - u^0(s)]ds\right\|_{\mathbb{H}}^p, \\ &\leq 3^{p-1}M^pT^{p-1}\int_0^t \mathbb{E}\|C(s)[u^n(s) - u^0(s)]\|_{\mathbb{H}}^p ds,\end{aligned}$$

and

$$\begin{aligned}\vartheta_n^2(t) &= 3^{p-1}\mathbb{E}\left\|\int_0^t R(t-s)[\sigma_n(s) - \bar{\sigma}(s)]ds\right\|_{\mathbb{H}}^p, \\ &\leq 3^{p-1}M^pT^{p-1}\int_0^t \mathbb{E}\|\sigma_n(s) - \bar{\sigma}(s)\|_{\mathbb{H}}^p ds.\end{aligned}$$

Since  $C$  is a compact operator, we have

$$\int_0^t \mathbb{E}\|C(s)[u^n(s) - u^0(s)]\|_{\mathbb{H}}^p ds \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and by Lebesgue's dominated convergence theorem,

$$\int_0^t \mathbb{E}\|\sigma_n(s) - \bar{\sigma}(s)\|_{\mathbb{H}}^p ds \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Consequently,

$$\vartheta_n^1(t), \vartheta_n^2(t) \rightarrow 0 \text{ as } n \rightarrow \infty.$$



On the other hand, by **(A<sub>5</sub>)** and Hölder inequality, we obtain

$$\begin{aligned}
 \vartheta_n^3(t) &= 3^{p-1} \mathbb{E} \left\| \int_0^t R(t-s) [f(s, \bar{x}_s^n) - f(s, \bar{x}_s)] dw(s) \right\|_{\mathbb{H}}^p \\
 &\leq 3^{p-1} C_p \left[ \int_0^t \left[ \|R(t-s)\|_{\mathbb{H}}^p \mathbb{E} \|f(s, \bar{x}_s^n) - f(s, \bar{x}_s)\|_{\mathbb{H}}^p \right]^{2/p} ds \right]^{p/2} \\
 &\leq 3^{p-1} C_p M^p T^{p/2-1} \int_0^t \mathbb{E} \|f(s, \bar{x}_s^n) - f(s, \bar{x}_s)\|_{\mathbb{H}}^p ds \\
 &\leq 3^{p-1} C_p M^p T^{p/2-1} L_f \int_0^t \|\bar{x}_s^n - \bar{x}_s\|_{\mathcal{B}}^p ds \\
 &\leq 6^{p-1} C_p K_T^p M^p T^{p/2} L_f \sup_{s \in [0, T]} \mathbb{E} \|\bar{x}^n(s) - \bar{x}(s)\|_{\mathbb{H}}^p \\
 &= 6^{p-1} C_p K_T^p M^p T^{p/2} L_f \sup_{s \in [0, T]} \mathbb{E} \|x^n(s) - \tilde{x}(s)\|_{\mathbb{H}}^p \\
 &\quad \text{since } \bar{x} = x \text{ on } J, \\
 &= \theta \|x^n - \tilde{x}\|_{\mathcal{C}}^p
 \end{aligned}$$

where  $\theta = 6^{p-1} C_p K_T^p M^p T^{p/2} L_f$ . Then, we have

$$\mathbb{E} \|x^n(t) - \tilde{x}\|_{\mathbb{H}}^p \leq \vartheta_n^1(t) + \vartheta_n^2(t) + \theta \|x^n - \tilde{x}\|_{\mathcal{C}}^p,$$

this implies that

$$\|x^n - \tilde{x}\|_{\mathcal{C}}^p \leq \frac{\vartheta_n^1(t) + \vartheta_n^2(t)}{1 - \theta},$$

since the right-hand of the above inequality tends to 0 as  $n \rightarrow \infty$ , we deduce that

$$x^n \rightarrow \tilde{x} \text{ in } \mathcal{BC} \text{ as } n \rightarrow \infty.$$

Moreover, by **(A<sub>4</sub>)**, we have

$$\sigma_n(\cdot) \rightarrow \sigma(\cdot, \tilde{x}) \text{ in } \mathcal{BC} \text{ as } n \rightarrow \infty.$$

By the uniqueness of limit, we obtain

$$\bar{\sigma}(t) = \sigma(t, \tilde{x}) \text{ for all } t \in J.$$

Thus,  $\tilde{x}$  takes the following form

$$\begin{aligned}
 \tilde{x}(t) &= R(t)\varphi(0) + \int_0^t R(t-s)C(s)u^0(s)ds \\
 &\quad + \int_0^t R(t-s)\sigma(s, \tilde{x}_s)ds + \int_0^t R(t-s)f(s, \tilde{x}_s)dw(s), \quad t \in J,
 \end{aligned}$$

which is just a mild solution of system (1) corresponding to  $u^0$ . Since  $\mathcal{BC} \hookrightarrow L^1(J, \mathbb{H})$ , using the assumptions **(L1)-(L4)** and Balder's theorem (see [3]), we get

$$\begin{aligned}
 \epsilon &= \lim_{n \rightarrow \infty} \mathbb{E} \int_0^T L(t, x_t^n, x^n(t), u^n(t))dt \\
 &\geq \mathbb{E} \int_0^T L(t, \tilde{x}_t, \tilde{x}(t), u^0(t))dt \geq \mathcal{J}(\tilde{x}, u^0) \geq \epsilon.
 \end{aligned}$$

Finally, we conclude that  $\mathcal{J}$  attains its minimum at  $(\tilde{x}, u^0) \in \mathcal{BC} \times \mathbb{U}_{ad}$ . The proof is complete.  $\square$

## 5. EXAMPLE

In this section, we illustrate the obtained theory. We consider the following stochastic integrodifferential equation with infinite delay of the form:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} z(t, \rho) = \left[ \frac{\partial^2}{\partial \rho^2} z(t, \rho) + \int_0^t \alpha(t-s) \frac{\partial^2}{\partial \rho^2} z(s, \rho) ds + \int_{-\infty}^0 a_1(t, s-t, \rho, z(s, \rho)) ds \right. \\ \quad \left. + \int_{[0, \pi]} \int_0^T q(t, \rho, u(s, \tau)) ds d\tau \right] dt + \int_{-\infty}^0 a_2(t) a_3(s-t) z(s, \rho) ds d\omega(t), \\ \quad t \in [0, T], \rho \in [0, \pi], \\ z(t, 0) = z(t, \pi) = 0, \quad t \in [0, T], \\ z(\tau, \rho) = z_0(\tau, \rho), \quad -\infty \leq \tau \leq 0, \quad \rho \in [0, \pi], \end{array} \right. \quad (15)$$

where  $\omega(t)$  denotes a one dimensional standard Wiener process defined on a stochastic space  $(\Omega, \mathcal{F}, \mathbb{P})$  and the cost function is given by

$$\begin{aligned} \mathcal{J}(z, u) = & \mathbb{E} \int_0^T \left[ \int_{[0, \pi]} \int_{-\infty}^0 |z^u(t+s, \rho)|^2 ds d\rho + \int_{[0, \pi]} |z^u(t, \rho)|^2 d\rho \right. \\ & \left. + \int_{[0, \pi]} |u(t, \rho)|_{\mathbb{Y}}^2 d\rho \right] dt. \end{aligned}$$

Let  $\mathbb{H} = \mathbb{Y} = L^2([0, \pi])$  with the norm  $\|\cdot\|$  and define the operator  $A : D(A) \subseteq \mathbb{H} \rightarrow \mathbb{H}$  by

$$\left\{ \begin{array}{l} Av = \frac{\partial^2}{\partial \rho^2} v \\ D(A) = H_0^1([0, \pi]) \cap H^2([0, \pi]). \end{array} \right.$$

Then,

$$Av = - \sum_{n=1}^{\infty} n^2 \langle v, e_n \rangle e_n, \quad v \in D(A),$$

where  $e_n(\rho) = \sqrt{\frac{2}{\pi}} \sin(n\rho)$ ,  $n = 1, 2, \dots$  is the orthogonal set of eigenvectors of  $A$ .

It is well known that  $A$  is the infinitesimal generator of a strongly continuous semigroup on  $\mathbb{H}$ , thus  $(\mathbf{R}_1)$  is true.

Let  $B : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$  be the operator defined by

$$B(t)y = \alpha(t)Ay, \text{ for } t \geq 0 \text{ and } y \in D(A).$$

The control  $u$  belongs, for a constant  $\eta > 0$ , to the following set

$$\begin{aligned} \mathbb{U}_{ad} = & \left\{ u(\cdot, y) : J \rightarrow \mathbb{Y} \text{ measurable, } \mathcal{F}_T\text{-adapted stochastic processes} \right. \\ & \left. \text{and } \|u\|_{L_{\mathcal{F}}^p(J, \mathbb{Y})} \leq \eta \right\}. \end{aligned}$$

Let us consider  $r > 0$ ,  $1 \leq q < \infty$  and let  $h : (-\infty, -r] \rightarrow \mathbb{R}$  be a non-negative measurable function which satisfies the conditions  $(h-5)$  and  $(h-6)$  in the terminology of Hino et al. [8].

Briefly, this means that  $h$  is locally integrable and there is a non-negative, locally bounded function  $G$  on  $(-\infty, 0]$  such that  $h(\xi + \tau) \leq G(\xi)h(\tau)$  for all  $\xi \leq 0$  and  $\tau \in (-\infty, -r) \setminus N_\xi$  where  $N_\xi \subset (-\infty, -r)$  is a set whose Lebesgue measure on zero.

We denote by  $\mathcal{C}_r \times L^q(h, \mathbb{H})$  the set of all functions

$\varphi : (-\infty, -r) \rightarrow \mathbb{H}$  such that  $\varphi|_{[-r, 0]} \in \mathcal{C}([-r, 0], \mathbb{H})$ ,  $\varphi(\cdot)$  is Lebesgue measurable on  $(-\infty, -r)$  and  $h\|\varphi\|^q$  is Lebesgue integrable in  $(-\infty, -r)$ . The seminorm is given by

$$\|\varphi\|_{\mathcal{B}} = \sup_{-r \leq \tau \leq 0} \|\varphi(\tau)\| + \left( \int_{-\infty}^{-r} h(\tau) \|\varphi(\tau)\|^q d\tau \right)^{\frac{1}{q}}.$$

We notice that Axioms **(A)-(C)** are satisfied by the space  $\mathcal{B} = \mathcal{C}_r \times L^q(h, \mathbb{H})$ .

Moreover when,  $r = 0$  and  $q = 2$ , we can take  $h = 1$ ,  $M(t) = G(-t)^{\frac{1}{2}}$  and  $K(t) = 1 + \left( \int_{-t}^0 h(\tau) d\tau \right)^{\frac{1}{2}}$ , for  $t \geq 0$ .

In addition, we assume that:

- (i): The function  $a_1 : \mathbb{R}^4 \rightarrow \mathbb{R}$  is completely continuous and there exists a continuous function  $\zeta : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$|a_1(t, s, \rho, y)| \leq \zeta(t, s) |y|, \quad (t, s, \rho, y) \in \mathbb{R}^4.$$

- (ii): The functions  $a_2, a_3 : \mathbb{R} \rightarrow \mathbb{R}$  are continuous, and  $d_c = \left( \int_{-\infty}^0 \frac{(a_3(s))^2}{h(s)} ds \right)^{\frac{1}{2}} < \infty$ .

- (iii): The function  $q : [0, T] \times [0, \pi] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there exists a continuous function  $q_e : [0, T] \times [0, \pi] \rightarrow \mathbb{R}$  such that :

$$|q(t, \rho, u)| \leq q_e(t, \rho) |u|$$

for all  $(t, \rho) \in [0, T] \times [0, \pi]$  and  $u \in L^2([0, T] \times [0, \pi])$ .

We take  $\varphi \in \mathcal{B} = \mathcal{C}_0 \times L^2(h, \mathbb{H})$  with  $\varphi(s)(\xi) = \varphi(s, \xi)$ . The nonlinear functions  $f : [0, T] \times \mathcal{B} \rightarrow \mathbb{H}$ ,  $\sigma : [0, T] \times \mathcal{B} \rightarrow \mathcal{L}_b$  are defined by

$$f(t, \phi)(\rho) = \int_{-\infty}^0 a_2(t) a_3(s) \phi(s)(\rho) ds,$$

$$\sigma(t, \phi)(\rho) = \int_{-\infty}^0 a_1(t, s, \rho, \phi(s)(\rho)) ds.$$

For all  $u \in L^2([0, T] \times [0, \pi])$ , we define an operator  $C$  as follows:

$$(Cu)(t, \rho) = \int_{[0, \pi]} \int_0^T q(t, \rho, u(s, \tau)) ds d\tau.$$

Using these definition, we can represent the system (15) in the abstract form :

$$\begin{cases} z'(t) = \left[ Az(t) + \int_0^t B(t-s)z(s, \rho) ds + (Cu)(t, \rho) \right. \\ \quad \left. + \sigma(t, \phi)(\rho) \right] dt + f(t, \phi)(\rho) d\omega(t) \text{ for } t \geq 0, \\ z(t, 0) = z(t, \pi) = 0 \quad t \in [0, T], \end{cases} \quad (16)$$

with the cost function

$$\mathcal{J}(z^u, u) = \mathbb{E} \int_0^T \left[ \int_{[0, \pi]} \|z^u(t+s)\|^2 ds + \|u(t)\|_{\mathbb{Y}}^2 + \|z^u(t)\|^2 \right] dt.$$

We suppose that  $\alpha$  is a bounded and  $C^1$  function such that  $\alpha'$  is bounded and uniformly continuous, which implies that the operator  $B(t)$  satisfies assumption  $(\mathbf{R}_2)$ . Consequently, we deduce the existence of resolvent operator  $(R(t))_{t \geq 0}$  for Eq. (2). From (i), we have,

$$\begin{aligned} \mathbb{E} \|\sigma(t, \phi)\|^p &= \mathbb{E} \left[ \left( \int_0^\pi \left| \int_{-\infty}^0 a_1(t, s, x, \phi(s)(x)) ds \right|^2 dx \right)^{\frac{1}{2}} \right]^p \\ &\leq \mathbb{E} \left[ \left( \int_0^\pi \left( \int_{-\infty}^0 |\zeta(t, s) \phi(s)(x)| ds \right)^2 dx \right)^{\frac{1}{2}} \right]^p \\ &\leq \mathbb{E} \left[ \left( \int_{-\infty}^0 \frac{(\zeta(t, s))^2}{h(s)} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^0 h(s) \|\phi(s)\|^2 ds \right)^{\frac{1}{2}} \right]^p \\ &\leq c_1(t) \left[ \|\phi(0)\| + \left( \int_{-\infty}^0 h(s) \|\phi(s)\|^2 ds \right)^{\frac{1}{2}} \right]^p \\ &\leq c_1(t) \|\phi\|_{\mathcal{B}}^p \end{aligned}$$

for all  $(t, \phi) \in [0, T] \times \mathcal{B}$ , where  $c_1(t) = \left[ \left( \int_{-\infty}^0 \frac{(\zeta(t, s))^2}{h(s)} ds \right)^{\frac{1}{2}} \right]^p$ .

By (ii), when we take again  $\Theta_\sigma(s) = s$ ,  $\int_1^{+\infty} \frac{1}{\Theta_\sigma(s)} ds = \infty$ , we obtain

$$\begin{aligned} \mathbb{E} \|f(t, \phi) - f(t, \phi_1)\|^p &= \mathbb{E} \left[ \left( \int_0^\pi \left| \int_{-\infty}^0 a_2(t) a_3(s) [\phi(s)(x) - \phi_1(s)(x)] ds \right|^2 dx \right)^{\frac{1}{2}} \right]^p \\ &\leq \mathbb{E} \left[ \|a_2\|_\infty \left( \int_0^\pi \left( \int_{-\infty}^0 a_3(s) |\phi(s)(x) - \phi_1(s)(x)| ds \right)^2 dx \right)^{\frac{1}{2}} \right]^p \\ &\leq \mathbb{E} \left[ \|a_2\|_\infty \left( \int_{-\infty}^0 \frac{(a_3(s))^2}{h(s)} ds \right)^{\frac{1}{2}} \left( \int_{-\infty}^0 h(s) \|\phi(s) - \phi_1(s)\|^2 ds \right)^{\frac{1}{2}} \right]^p \\ &\leq \tilde{L}_f \left[ \|\phi(0)\| + \left( \int_{-\infty}^0 h(s) \|\phi(s) - \phi_1(s)\|^2 ds \right)^{\frac{1}{2}} \right]^p \\ &\leq \tilde{L}_f \|\phi - \phi_1\|_{\mathcal{B}}^p \end{aligned}$$

for all  $(t, \phi), (t, \phi_1) \in [0, T] \times \mathcal{B}$ , where  $\tilde{L}_f = [\|a_2\|_\infty d_c]^p$  with  $d_c = \left( \int_{-\infty}^0 \frac{(a_3(s))^2}{h(s)} ds \right)^{\frac{1}{2}}$ .

Therefore, the assumptions  $(\mathbf{A}_1) - (\mathbf{A}_5)$  are satisfied. By assumption (iii), it is obvious that  $(Cu)(t, \rho)$  is measurable in  $[0, t] \times [0, \pi]$ . For  $u \in L^2([0, T] \times [0, \pi])$  we have

$$\begin{aligned}
\int_{[0,\pi]} \int_0^T |(Cu)(t,x)|^2 dt dx &= \int_{[0,\pi]} \int_0^T \left| \int_{[0,\pi]} \int_0^T q(t,x,u(s,\tau)) ds d\tau \right|^2 dt dx \\
&\leq \int_{[0,\pi]} \int_0^T \left[ \int_{[0,\pi]} \int_0^T |q_e(t,x)|^2 ds d\tau \int_{[0,\pi]} \int_0^T |u(s,\tau)|^2 ds d\tau \right] dt dx \\
&\leq (M_{q_e} \pi T)^2 \int_{[0,\pi]} \int_0^T |u(s,\tau)|^2 ds d\tau
\end{aligned}$$

where  $M_{q_e} = \max_{(t,x) \in [0,T] \times [0,\pi]} |q_e(t,x)|$ . This implies that the operator  $C : L^2([0,T] \times [0,\pi]) \rightarrow L^2([0,T] \times [0,\pi])$ , and

$$\|Cu\|_{L^2([0,T] \times [0,\pi])} \leq M_{q_e} \pi T \|u\|_{L^2([0,T] \times [0,\pi])}.$$

Hence, we conclude that for all  $u \in L^2([0,T] \times [0,\pi])$ ,  $C$  is a compact operator in  $L^2([0,T] \times [0,\pi])$ . Additionally, all conditions of Theorem 4.1 hold, thus system (16) has at least one optimal pair.

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