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ON BIQUADRATIC FIELDS THAT ADMIT UNIT POWER INTEGRAL BASIS

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Dedicated to Kálmán Győry and András Sárközy on the occasion of their 70th birthday

Abstract. We consider biquadratic number fields whose maximal orders have power integral bases consisting of units. We prove an effective and efficient criteria to decide whether the maximal order of a biquadratic field has a unit power integral basis or not. In particular we can determine all trivial biquadratic fields whose maximal orders have a unit power integral basis.

1. Introduction

In general it is a hard problem to decide which number fields $\mathbb{K}$ admit power integral bases, PIB for short. In general to decide whether a number field $\mathbb{K}$ admits a PIB or not leads to so called index form equations. These index form equations have been treated by several authors and in particular by Győry (see e.g. [16,17]). Therefore it was possible to develop algorithms which find PIB for concrete types of fields. For instance the problem was solved for cubic [14] and quartic [13] and also for fields of higher degree of

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special form (see e.g. [10,12]). For a detailed overview one may have a look into Gaál's book [11] on this topic.

More recently Narkiewicz and Jarden [18] raised the following problem. Which number fields have the property that their maximal orders are generated by units. This problem was considered for several kinds of fields, e.g. for quadratic [1,3], cubic [21], pure quartic [9] and biquadratic fields [24].

With these two topics in mind the following question arises.

PROBLEM 1. *For which number fields does there exist a power integral basis consisting of units?*

The idea to solve this problem is quite simple. Let $\mathbb{K}$ be a number field of degree n and $\mathbb{Z}_{\mathbb{K}}$ be the maximal order of $\mathbb{K}$, furthermore assume $\mathcal{B} = \{1, \beta_2, \ldots, \beta_n\}$ is an integral basis of $\mathbb{Z}_{\mathbb{K}}$ and let $\eta = x_1 + x_2\beta_2 + \cdots + x_n\beta_n \in \mathbb{Z}_{\mathbb{K}}$. Then we consider the equation

$$I(\eta) = \big[\mathbb{Z}_{\mathbb{K}} : \mathbb{Z}[\theta]\big] = F(x_2, \ldots, x_n) = \pm 1 \tag{1}$$

which is solvable if and only if $\{1, \theta, \ldots, \theta^{n-1}\}$ is a PIB of $\mathbb{K}$. Now assume $\theta \in \mathbb{Z}_{\mathbb{K}}^*$ is a solution to the index equation (1). Then also θ^{-1} is a solution to (1). Beside "trivial" cases the solutions corresponding to θ and θ^{-1} should be distinct. Recently this idea has been successfully applied by the second author [23] in the case of pure quartic fields $\mathbb{K} = \mathbb{Q}\big[\sqrt[4]{m}\,\big]$. He characterized all orders $\mathbb{Z}\big[\sqrt[4]{m}\,\big]$ which admit UPIB. In this case the corresponding index form equation can be transformed into the well known Diophantine equation

$$X^2 - 4mY^4 = 1. \tag{2}$$

The aim of this paper is to apply this idea to biquadratic fields, i.e. quartic fields that have a Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore the following convention will be useful.

DEFINITION 1. We call the biquadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{dn}, \sqrt{dm}\,)$ given in canonic form if:

(1) d, m, n are square-free integers which are relatively prime such that m and n are odd and $dm \neq 1$, $dn \neq 1$ and $mn \neq 1$;

(2) $dm \equiv dn \bmod 4$;

(3) $d > 0$, $m > n$ and $d \leqq \min\{|m|, |n|\}$ if $dm \equiv dn \equiv 1 \bmod 4$. Moreover we put $\delta \in \{0, 1\}$ according to $mn \equiv (-1)^{\delta} \bmod 4$.

We call the field $\mathbb{K}$ trivial if $d = 2^{\delta}$, $|m| = 1$ or $|n| = 1$.

With these conventions Gras and Tanoé [15] showed that $\mathbb{K}$ admits a PIB is equivalent to the solubility of the system of Pell equations

$$z^2 2^{\delta} m - y^2 2^{\delta} n = 4s, \quad z^2 2^{-\delta} d - x^2 2^{\delta} n = s, \quad y^2 2^{-\delta} d - x^2 2^{\delta} m = s, \tag{3}$$

where $s \in \{\pm 1\}$. Note that every solution of a single Pell equation in (3) can be obtained in the form $2\big(y_0\sqrt{2^\delta m} + z_0\sqrt{2^\delta n}\,\big)^k = 2\varepsilon^k$ or $\big(y_0\sqrt{2^{-\delta}d} + x_0\sqrt{2^\delta n}\,\big)^j = \eta^j$ or $\big(z_0\sqrt{2^{-\delta}d} + x_0\sqrt{2^\delta m}\,\big)^l = \rho^l$. We call a solution with $|k| = 1$ or $|j| = 1$ or $|l| = 1$ a fundamental solution (in the first case one should also ignore the factor 2). Using the idea presented above we are able to show the following result.

THEOREM 1. *Let $\mathbb{K}$ be a biquadratic number field and let ε be the fundamental solution of one of the Pell equations in* (3) (*cf. Lemma* 2). *If there exists no solution to* (3) *that is implied by ε^k with $1 \leqq k \leqq 27$, then $\mathbb{K}$ admits no UPIB or there exists $a \in \mathbb{Z}$ such that $\mathbb{Z}[\theta] = \mathbb{Z}_\mathbb{K}$ with $\theta \in \mathbb{Z}_\mathbb{K}^*$ and*

$$\theta = a + x\frac{1 - \delta + 2^\delta\sqrt{mn}}{2} + \frac{y - z}{2}\sqrt{dn} + z\frac{\sqrt{dn} + \sqrt{dm}}{2},$$

where (x, y, z) is a solution to (3) *with $xyz = 0$.*

This theorem will help us to find all trivial biquadratic fields that provide a UPIB. In particular, we are able to prove

THEOREM 2. *Let $\mathbb{K} = \mathbb{Q}(\sqrt{dm}, \sqrt{dn})$ be a trivial biquadratic field. Then $\mathbb{K}$ has a UPIB if and only if one of the two cases holds*:

- *$d = 1$, $m = n + 4$, $\delta = 0$ and $n \equiv 3 \bmod 4$, n and $n + 4$ are square-free and*

$$\mathbb{Z}_\mathbb{K} = \mathbb{Z}\left[\frac{\sqrt{n} + \sqrt{n+4}}{2}\right];$$

- *$d = 2$, $m = n + 2$, $\delta = 1$ and n odd, n and $n + 2$ are square-free and*

$$\mathbb{Z}_\mathbb{K} = \mathbb{Z}\left[\frac{\sqrt{2n} + \sqrt{2n+4}}{2}\right].$$

In particular, we can show that the maximal orders of the family

$$\mathbb{K} = \mathbb{Q}\left(\sqrt{n}, \sqrt{\frac{n-1}{4}}\right)$$

of trivial biquadratic fields do not have UPIBs:

COROLLARY 1. *Let $n \equiv 1 \bmod 4$ and assume n and $\frac{n-1}{4}$ are square-free. Then the field $\mathbb{K} = \mathbb{Q}\left(\sqrt{n}, \sqrt{\frac{n-1}{4}}\right)$ admits no UPIB.*

Note that for all classes of trivial biquadratic fields there exist infinitely many n for which the assumptions of Theorem 2 as well as Corollary 1 hold (cf. [8]).

This corollary is representative for a whole class of examples of parameterized families of biquadratic fields. In view of Theorem 1 we only have to find the fundamental solution of one single Pell equation in (3). In the last section we will treat a further example.

Our paper is organized as follows. In the next two sections we present useful results on maximal orders and power integral bases of biquadratic fields (Section 2) and results on the structure of solutions to Pell equations of the form $ax^2 - by^2 = \pm 1, \pm 4$ (Section 3). Using ideas due to Bennett, Cipu, Mignotte and Okazaki [4,5,7] we will show that the system of Pell equations (3) has at most two solutions provided there exists no small solution to (3) (Sections 4 and 5). Moreover, we show that these two solutions are closely connected by the following relation. Let the first solution be induced by ε^{k_1} and the second solution be induced by ε^{k_2}, with $k_1 < k_2$, then we have $k_1 | k_2$. On the other hand if $\theta \in \mathbb{Z}_{\mathbb{K}}^*$ induces a UPIB also θ^{-1} induces a UPIB, hence in case of $\mathbb{K}$ admits a UPIB we have two solutions to (3). In Section 6 we show that the two solutions coming from θ and θ^{-1} cannot satisfy a relation of the form $k_1 | k_2$, hence a contradiction to $\mathbb{K}$ admits a UPIB. In the last two sections we apply Theorem 1 to two examples. In the first example we classify all trivial biquadratic fields, i.e. $d = 2^\delta$ or $|m| = 1$ or $|n| = 1$, that admit a UPIB (see also Corollary 1). In the last section we consider the infinite family $\mathbb{K} = \mathbb{Q}(\sqrt{18n^2 + 17n + 4}, \sqrt{2n^2 + n})$ of biquadratic fields and apply Theorem 1.

2. Maximal orders and power integral bases

This section is devoted to the results due to Gras and Tanoè [15]. We start with a result due to Williams [22] on the integral basis of $\mathbb{K}$. In view of our canonic form of $\mathbb{K}$ we state William's result in the following form:

LEMMA 1. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{dm}, \sqrt{dn})$ *be given in canonic form, then two cases may occur*:

(1) *Assume that* $dm \equiv dn \equiv 1 \bmod 4$ *and assume that we choose* $\lambda = \pm 1$ *such that* $d \equiv m \equiv n \equiv \lambda \bmod 4$. *Then*

$$\mathcal{B}_{\mathbb{K}} = \left\{1, \frac{1 + \sqrt{mn}}{2}, \frac{1 + \sqrt{dn}}{2}, \frac{1 + \lambda\sqrt{mn} + \sqrt{dn} + \sqrt{dm}}{4}\right\}$$

is an integral basis and $d_{\mathbb{K}} = (dmn)^2$ *is the field discriminant.*

(2) *Assume that* $dm \equiv dn \equiv 2$ *or* $3 \bmod 4$. *Then*

$$\mathcal{B}_{\mathbb{K}} = \left\{1, \frac{1 - \delta + 2^\delta\sqrt{mn}}{2}, \sqrt{dn}, \frac{\sqrt{dn} + \sqrt{dm}}{2}\right\}$$

is an integral basis and $d_{\mathbb{K}} = \left(2^{\delta+2}dmn\right)^2$ *is the field discriminant.*

With the notations of Lemma 1 we can state now the result due to Gras and Tanoé [15].

PROPOSITION 1. *Let* $\mathbb{K} = \mathbb{Q}(\sqrt{dm}, \sqrt{dn})$ *be given in canonic form. If* $dm \equiv dn \equiv 1 \bmod 4$, *then* $\mathbb{K}$ *admits no PIB and hence no UPIB. In the other cases* $\mathbb{K}$ *admits a PIB if and only if* (3) *has a solution, say* (x, y, z). *In this case* $\{1, \alpha, \alpha^2, \alpha^3\}$ *is a PIB, where* α *has coordinates* $\left(x, \frac{y-z}{2}, z\right)$ *with respect to the integral basis* $\mathcal{B}_{\mathbb{K}}$. *A necessary condition for* (3) *to be solvable is that* $2^{\delta}m = 2^{\delta}n + 2^{2-\delta}d$.

By the linear relation of d, m, n in Proposition 1 we see that either n/m, $4^{\delta}n/d$ or $4^{\delta}m/d$ lie in the open interval $(1/8, 8)$. Therefore it is always possible to find for a given Pell equation in (3) a second Pell equation in (3) such that the resulting system of Pell equations is of the form

$$aX^2 - bY^2 = \pm 4^{e_1}, \quad cX^2 - dZ^2 = \pm 4^{e_2},$$

with $1/8 < a/c < 8$, $e_1, e_2 \in \{0, 1\}$ and fixed signs.

3. Pell equations

Since (3) we are interested in Pell equations of the type

$$ax^2 - by^2 = \pm 4^e, \tag{4}$$

where $e \in \{0, 1\}$ and $0 < a < b$ are integers such that ab is square-free. Note that by Definition 1 we have $d > 0$. If $n < 0$ then (3) is very easily solvable, which justifies our assumption. First, let us consider the structure of solutions of Pell equations of type (4).

LEMMA 2. *There exist non-zero integers* x_0, y_0 *such that for each solution* (x, y) *to* (4) *we have*

$$x\sqrt{a} + y\sqrt{b} = 2^e \left(\frac{x_0\sqrt{a} + y_0\sqrt{b}}{2^e} \right)^k = 2^e \alpha^k.$$

Moreover, if $a > 1$, *then* k *has to be odd.*

Note that the algebraic integer α is a unit in $\mathbb{K}$ and will be called the fundamental solution to (4). Although the lemma seems to be known, we have not found an adequate reference. Therefore we give here a short proof.

PROOF. If $a = 1$, then we have nothing to prove because (4) turns into a usual Pell equation. Thus we assume in the sequel $a > 1$. We multiply (4) by a and obtain

$$(ax)^2 - aby^2 = \pm 4^e a. \tag{5}$$

We distinguish now the three cases 1) $e = 0$, 2) $e = 1$ and $ab \not\equiv 1 \bmod 4$ and 3) $e = 1$ and $ab \equiv 1 \bmod 4$.

Let us consider the first case. Since every prime p with $p|a$, also divides the discriminant of $\mathbb{Q}(\sqrt{ab})$, we know by a theorem of Dirichlet that $(p) = \mathfrak{p}^2$ for some prime ideal $\mathfrak{p}$, thus $(a) = \mathfrak{a}^2$ for some ideal $\mathfrak{a}$. If $\mathfrak{a}$ is not a principal ideal, (5) is not solvable. Otherwise there exist $\tilde{x}_0$ and y_0 such that every solution (ax, y) to (5) can be written in the form $ax + y\sqrt{ab} = (\tilde{x}_0 + y_0\sqrt{ab})\,\varepsilon^k$, where $\varepsilon > 1$ is the fundamental unit of $\mathbb{Z}[\sqrt{ab}]$. Since every prime ideal dividing (a) also divides $ax + y\sqrt{ab}$, we conclude $\tilde{x}_0 = ax_0$, hence dividing the solution by $\sqrt{a}$ we obtain that every solution (x, y) to (4) is of the form

$$x\sqrt{a} + y\sqrt{b} = (x_0\sqrt{a} + y_0\sqrt{b})\,\varepsilon^k.$$

Obviously $\eta = x_0\sqrt{a} + y_0\sqrt{b}$ is also a unit (of $\mathbb{Q}(\sqrt{a}, \sqrt{b})$) such that $\eta^2 \in \mathbb{Z}[\sqrt{ab}]$ and hence $\eta^2 = \varepsilon^l$. In the case of l is even we can take square roots and find $\eta = \pm\varepsilon^{l/2} \in \mathbb{Q}(\sqrt{ab})$, a contradiction. Therefore we have $\eta^2 = \varepsilon^{2k}\varepsilon$, i.e.

$$\tilde{\eta}^2 = \left(\frac{\eta}{\varepsilon^k}\right)^2 = \varepsilon.$$

It is easy to see that $\tilde{\eta}$ is of the form $\bar{x}_0\sqrt{a} + \bar{y}_0\sqrt{b}$ and we have

$$x\sqrt{a} + y\sqrt{b} = (\bar{x}_0\sqrt{a} + \bar{y}_0\sqrt{b})\,\varepsilon^{\bar{k}} = \tilde{\eta}^{2\bar{k}+1}.$$

In the second case the proof runs analogously with the only exception that we have to consider the ideal $(4a)$ instead of (a). Since 2 is ramified in $\mathbb{Q}(\sqrt{ab})$, no new arguments have to be included, we only have to keep track of the factor 2.

In the third case note that all solutions (x, y) of the equation

$$x^2 - aby^2 = \pm 4$$

are of the form $x + y\sqrt{ab} = 2\varepsilon^k$, where $\varepsilon = \frac{x_0 + y_0\sqrt{ab}}{2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{ab})$. □

Next we want to estimate the growth of solutions.

LEMMA 3. *Assume that $a > 4$ and $k \geqq 28$ in Lemma 2. Then the inequalities*

$$x > \frac{x_0^k a^{(k-1)/2}}{2^k} \quad \text{and} \quad x > \frac{y_0^k b^{(k-1)/2}}{2^k} \tag{6}$$

hold. In any case (but still assuming $k \geqq 28$) we have

$$x > \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - 1}{\sqrt{5}} > 1.572^k. \tag{7}$$

PROOF. Using the notation of Lemma 2 we write $\alpha = \frac{x_0\sqrt{a}+y_0\sqrt{b}}{2^e}$. Assume that $x_0 a > 1$, then we obtain for a solution

$$x = \frac{2^{e-1}}{\sqrt{a}} \cdot \left(\alpha^k \pm \alpha^{-k}\right) > \frac{\left(\frac{x_0\sqrt{a}+1}{2}\right)^k - 1}{\sqrt{a}} \geqq \frac{x_0^k a^{(k-1)/2}}{2^k}.$$

The validity of the second inequality can be seen by considering the cases $e = 0$ and $e = 1$ separately. Of course the first statement of the lemma is still true for $x_0 a = 1$. Moreover, the second inequality can be established analogously.

On the other hand we know $\alpha \geqq \frac{1+\sqrt{5}}{2}$ in which case $e = 1$, hence

$$x \geqq \frac{\alpha^k \pm \alpha^{-k}}{\sqrt{5}} > \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - 1}{\sqrt{5}} > 1.572^k. \quad \square$$

The next lemma is due to Nagell [20].

LEMMA 4. *Let $1 < a < b$ be integers such that ab is square-free. Then at most one of the two equations*

$$ax^2 - by^2 = \pm 1$$

has a solution.

PROOF. The lemma is an immediate consequence of Nagell's result [20, Theorem 3, Part 1]. $\square$

4. Hypergeometric method

Suppose that the system of Pell equations (3) has two solutions corresponding to the exponents (k_1, j_1, l_1) and (k_2, j_2, l_2). In view of Theorem 1 we show that the second solution stays small with respect to the first solution. For this purpose we use the hypergeometric method and adept Bennett's method [4] to our case. We use the following effective irrationality measure (see [4]):

THEOREM 3. *If a_i, p_i, q and N are integers for $0 \leqq i \leqq 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \leqq j \leqq 2$, q nonzero and $0 < M^9 < N$, where*

$$M = \max_{0 \leqq i \leqq 2} \{\, |a_i| \,\},$$

then we have

$$\max_{0\leqq i\leqq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\}>(130N\Upsilon)^{-1}q^{-\lambda},$$

where

$$\lambda=1+\frac{\log(33N\Upsilon)}{\log\left(1.7N^2\prod_{0\leqq i<j\leqq 2}(a_i-a_j)^{-2}\right)}$$

and

$$\Upsilon=\begin{cases}\dfrac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & \text{if } a_2-a_1\geqq a_1-a_0,\\[2ex] \dfrac{(a_2-a_0)^2(a_1-a_1)^2}{a_1+a_2-2a_0} & \text{if } a_2-a_1<a_1-a_0.\end{cases}$$

We consider in more generality the system of Pell equations

$$ax^2-by^2=\delta_1,\qquad cx^2-dz^2=\delta_2, \tag{8}$$

with $\delta_i=\pm 4^{e_i}$ with $e_i\in\{0,1\}$ and the fixed signs for $i=1,2$ such that $a\delta_2\neq c\delta_1$. Then all solutions (see Lemma 2) of the first equation are of the form

$$x\sqrt{a}+y\sqrt{b}=2^{e_1}\left(\frac{x_0\sqrt{a}+y_0\sqrt{b}}{2^{e_1}}\right)^k=2^{e_1}\alpha^k$$

and all solutions of the second equation are of the form

$$x\sqrt{c}+z\sqrt{d}=2^{e_2}\left(\frac{x_0'\sqrt{c}+z_0\sqrt{b}}{2^{e_2}}\right)^j=2^{e_2}\beta^j$$

for some integers j, k. In view of Theorem 1 we may assume $j,k\geqq 28$.

PROPOSITION 2. *Assume that $\alpha<\beta$, $a/8<c<8a$ and $k_1\geqq 28$. Then we have*

$$\frac{k_2}{k_1}<\frac{7k_1^3+114.35k_1^2-1228.1k_1+656.97}{k_1(k_1-1)(k_1-27.11)}\cdot\frac{1}{1+10^{-12}}.$$

In particular if we assume that $k_1\geqq 28$ we find $k_2<311.492k_1$.

PROOF. Now let us assume that the system (8) has two positive solutions (x_i,y_i,z_i) with exponents k_i and j_i for $i=1,2$. We want to apply Theorem 3 and choose therefore $N=acx_1^2$, $a_1=\mp 4^{e_1}c$, $a_2=\mp 4^{e_2}a$, $p_1=bcy_1y_2$, $p_2=adz_1z_2$ and $q=acx_1x_2$.

First, we have to show that with this choice the assumption $N > M^9$ is satisfied. Let us assume that $a > c$. Then $M \leqq 4a$ and we obtain by Lemma 2

$$N > \frac{a^2}{8}x_1^2 > \frac{a^2}{8}\frac{x_0^{56}a^{27}}{2^{56}} > 4^9a^9 \geqq M^9,$$

provided $a > 14$. Similarly in the case $a \leqq c$ we obtain

$$N > \frac{c^2}{8}x_1^2 > \frac{c^2}{8}\frac{x_0^{56}c^{27}}{2^{56}} > 4^9c^9 \geqq M^9,$$

provided $c > 14$. If b or d is larger than $\max\{a, c\}$ we obtain even larger lower bounds for N by using the second inequality in Lemma 2. Therefore we may assume $14 \geqq \max\{a, b, c, d\}$, i.e. we are left to finitely many systems of Pell equations. For each sixtuple $(a, b, c, d, \delta_1, \delta_2)$ with $a\delta_2 \neq c\delta_1$ we show that there exists no solution such that $x > 1.572^{28}$ and

$$acx^2 \leqq \left(\max\left\{\,|\delta_2|a, |\delta_1|c\right\}\right)^9.$$

This can be done by a simple computer search. Since we did not find any such x we may assume that indeed $N > M^9$.

With the choice made above we obtain

$$\sqrt{1+\frac{a_1}{N}} - \frac{p_1}{q} = \frac{by_1}{ax_1}\left(\sqrt{\frac{a}{b}} - \frac{y_2}{x_2}\right)$$

and

$$\sqrt{1+\frac{a_2}{N}} - \frac{p_1}{q} = \frac{dz_1}{cx_1}\left(\sqrt{\frac{c}{d}} - \frac{z_2}{x_2}\right).$$

Next, we estimate $\frac{by_1}{ax_1}$:

$$\left(\frac{by_1}{ax_1}\right)^2 = \frac{b}{a}\left(1 + \frac{\pm 4^{e_1}}{ax_1^2}\right) \leqq 1.001\frac{b}{a},$$

hence

$$\frac{by_1}{ax_1} \leqq 1.01\sqrt{\frac{b}{a}}.$$

Moreover, we have

$$\left|\sqrt{\frac{a}{b}} - \frac{y_2}{x_2}\right| = \left|\frac{x_2\sqrt{a} - y_2\sqrt{b}}{x_2\sqrt{b}}\right| = \left|\frac{4^{e_1}}{\sqrt{b}x_1}\cdot\frac{1}{x_2\sqrt{a} - y_2\sqrt{b}}\right| \leqq \frac{4}{x_2^2\sqrt{ab}}.$$

Similar considerations for the second Pell equation of (8) yield

$$\max_{0\leqq i\leqq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\}<\frac{4.04}{x_2^2}\max\{1/a,1/c\}\leqq\frac{4.04}{x_2^2}. \tag{9}$$

Our next aim is to find a lower bound for the maximum. Therefore we use Theorem 3. Let us estimate the necessary quantities. We start with Υ. Let us note that Υ is maximal if a_2 and a_1 have opposite signs and are maximal. Without loss of generality we may assume that $a_2=4c<32a$ and $a_1=-4a$, i.e. $\Upsilon<\frac{2^{14}a^3}{9}$.

Next, consider λ. Assume that $ax_0>1$, then we obtain

$$\lambda\leqq 1+\frac{\log\left(\frac{11\cdot 2^{17}}{3}a^5x_1^2\right)}{\log\left(\frac{x_1^4}{2^{15}\cdot 3^4a^4}\right)}$$

$$\leqq 1+\frac{2k_1\left(\log x_0+\frac{\log a}{2}\right)+4\log a+13\log 2+\log(11/3)}{4k_1\left(\log x_0+\frac{\log a}{2}\right)-6\log a-23\log 2-\log 81}$$

$$\leqq 1+\frac{2k_1+17+\frac{\log(11/3)}{\log 2}}{4k_1-29-\frac{\log(81)}{\log 2}}=\frac{3k_1-8.23}{2k_1-17.67}.$$

The second equation is true since the function $\frac{2\log x+A}{4\log x-B}$ is decreasing provided A and B are positive. This is obvious in our case, and therefore we can insert the lower bounds for x_1 in order to get the stated upper bound. In the case of $ax_0=1$ we obtain the same expression for λ but $\log a=0$ and using the estimate $x_1>1.572^{k_1}$ we get

$$\lambda\leqq 1+\frac{\log\left(\frac{11\cdot 2^{17}}{3}x_1^2\right)}{\log\left(\frac{x_1^4}{2^{15}\cdot 3^4}\right)}\leqq 1+\frac{2k_1+\frac{\log\left(2^{17}\cdot 11\right)-\log 3}{\log 1.572}}{4k_1-\frac{\log\left(2^{15}\cdot 3^4\right)}{\log 1.572}}\leqq\frac{3k_1-1.88}{2k_1-16.35}$$

which yields a smaller upper bound for λ provided $k_1\geqq 9$, i.e. we may assume that

$$\lambda\leqq\frac{3k_1-8.23}{2k_1-17.67}. \tag{10}$$

Putting everything together Theorem 3 yields the lower bound

$$\frac{9x_1^{-2}}{65\cdot 2^{18}a^5}\cdot\left(2^3a^2x_1x_2\right)^{-\frac{3k_1-8.23}{2k_1-17.67}}. \tag{11}$$

Now comparing the lower (11) and the upper bound (9) for

$$\max_{0\leqq i\leqq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\}$$

yields

$$\frac{9x_1^{-2}}{65\cdot 2^{18}a^5}\cdot\left(2^3a^2x_1x_2\right)^{-\frac{3k_1-8.23}{2k_1-17.67}}<\frac{4.04}{x_2^2}.$$

Rearranging the above inequality we obtain

$$x_2^{k_1-27.11}<29.18^{2k_1-17.67}2^{45k_1-342.75}a^{16k_1-104.81}x_1^{7k_1-43.57}.$$

Taking logarithms and using (7) and (6) yields

$$\log(x_2)<\log(x_1)\left(\frac{14.92k_1-131.77}{k_1(k_1-27.11)}+\frac{68.96k_1-525.2}{k_1(k_1-27.11)}\right.$$

$$\left.+\frac{81.04k_1-530.82}{(k_1-1)(k_1-27.11)}+\frac{7k_1-43.57}{k_1-27.11}\right)\tag{12}$$

$$=\frac{7k_1^3+114.35k_1^2-1228.1k_1+656.97}{k_1(k_1-1)(k_1-27.11)}.$$

Note that this bound is still true for $a=1$ as well. Indeed in this case we would obtain an even sharper bound.

Now let us assume that $k_2=\sigma k_1$ for some $\sigma>1$. This yields

$$\log x_2-\sigma\log x_1=\log\left(2^{e_1-1}\right)-\sigma\log\left(2^{e_1-1}\right)+k_2\log\alpha-\sigma k_1\log\alpha$$

$$+\log\left(1+\alpha^{-2k_2}\right)-\sigma\log\left(1+\alpha^{-2k_1}\right)>-\frac{2\sigma}{\alpha^{2k_1}}.$$

Since we assume that $k_1\geqq 28$ we obtain

$$\log x_2>\log(x_1)\frac{k_2}{k_1}\left(1-\frac{2}{\alpha^{2k_1}\log x_1}\right)>\log(x_1)\frac{k_2}{k_1}\cdot\frac{1}{1+10^{-12}}.$$

Together with (12) this yields the proposition. □

5. Gap principle

The main purpose of this section is to prove the following proposition.

PROPOSITION 3. *Assume that* (8) *has no solution for* $1 \leqq k \leqq 27$. *Then* (8) *has at most two solutions and the corresponding exponents satisfy* $k_2 = qk_1$ *with* $q \leqq 311$.

The key point of the the proof of Proposition 3 is to show that a possible third solution is large with respect to the other two solutions, i.e. we establish a gap principle similar to those in [5,7].

Before we establish the gap principle we prove a divisibility property for exponents.

LEMMA 5. *Let* (k_1, j_1) *be the exponents of the smallest positive solution of system* (8) *and let* (k, j) *be the exponents of a further positive solution to* (8). *Then we have* $k_1|k$ *and* $j_1|j$.

The proof can be found in [7] (one has to combine there the first part of Lemma 5 and part (2) of Lemma 6) in the case of $e_1 = e_2 = 0$ in (8). In case of $e_1 = 1$ or $e_2 = 1$ the proof [7, Lemma 5] works as well.

Now let us turn to the gap principle.

LEMMA 6. *Assume* (8) *has three positive solutions with exponents* (k_i, j_i) *and* $1 \leqq i \leqq 3$ *with* $28 \leqq j_1 < j_2 < j_3$. *Further, assume that* $\frac{1}{8} < \frac{a}{c} < 8$. *Then we have*

$$j_3 - j_2 > 337 j_1 k_1 \geqq 337 j_1.$$

PROOF. The idea of the proof is similar to that used in the proof of [5, Lemma 2.2]. First, note that (in the notation of Section 3)

$$x = 2^{e_1 - 1} \frac{\alpha^k \pm \alpha^{-k}}{\sqrt{a}} = 2^{e_2 - 1} \frac{\beta^j \pm \beta^{-j}}{\sqrt{c}},$$

where the signs depend on the norm of the fundamental solutions. Therefore we have to consider all four possibilities. Now let us write $\alpha^k = e^u$ and $\beta^j = e^t$, where e denotes the base of the natural logarithm. Then the points $(u_i, t_i) = (k_i \log \alpha, j_i \log \beta)$ for $i = 1, 2, 3$ lie on the curve

$$F(u,t) = (e^u \pm e^{-u}) - (e^t \pm e^{-t}) \sqrt{\frac{a 4^{e_2 - e_1}}{c}} = 0.$$

Since we assume $j \geqq 28$ we have $u, t \geqq 28 \cdot \log\left(\left(1 + \sqrt{5}\right)/2\right) > 13.4$. It is easy to see that neither $F_u = 0$ nor $F_t = 0$ for $u, t > 13.4$. Therefore u is a differentiable function of t on this domain and we may differentiate implicitly. We obtain

$$\frac{du}{dt} = -\frac{F_t}{F_u} = \frac{e^t \mp e^{-t}}{(e^u \mp e^{-u}) \sqrt{\frac{a 4^{e_2 - e_1}}{c}}} \tag{13}$$

and

$$\frac{d^2u}{dt^2} = -\left(\frac{du}{dt}\right)^2 F_{uu} + 2\frac{du}{dt}F_{ut} - F_{tt} = \left(1 - \left(\frac{du}{dt}\right)^2\right)\frac{e^u \pm e^{-u}}{e^u \mp e^{-u}}. \tag{14}$$

Our aim is to estimate (14). Therefore we consider (13) first. Let us note that

$$e^t = (e^u \mp e^{-u})\sqrt{\frac{a4^{e_2-e_1}}{c}} \pm 2e^{-u}\sqrt{\frac{a4^{e_2-e_1}}{c}} \mp e^{-t},$$

hence $e^t = e^u\sqrt{\frac{a4^{e_2-e_1}}{c}} \cdot (1 + \theta_1 0.001)$, where $|\theta_1| < 1$ and therefore we obtain

$$e^t = e^u\sqrt{\frac{a4^{e_2-e_1}}{c}}\left(1 \pm e^{-2u}\sqrt{\frac{a4^{e_2-e_1}}{c}} \mp e^{-t-u}\right) \tag{15}$$

$$= e^u\sqrt{\frac{a4^{e_2-e_1}}{c}}\left(1 + \frac{5.85\theta_2}{e^{2u}}\right),$$

where $|\theta_2| < 1$. The last inequality is a consequence of the fact that $a/c, c/a < 8$. Now we obtain

$$\frac{du}{dt} = \frac{(e^u \mp e^{-u})\sqrt{\frac{a4^{e_2-e_1}}{c}} \pm 2e^{-u}\sqrt{\frac{a4^{e_2-e_1}}{c}} \mp 2e^{-t}}{(e^u \mp e^{-u})\sqrt{\frac{a4^{e_2-e_1}}{c}}}$$

$$= 1 + \frac{\pm 2e^{-u}\sqrt{\frac{a4^{e_2-e_1}}{c}} \mp 2e^{-u}\sqrt{\frac{c}{a4^{e_2-e_1}}} \cdot \frac{1}{1+\frac{5.85\theta_2}{e^{2u}}}}{(e^u \mp e^{-u})\sqrt{\frac{a4^{e_2-e_1}}{c}}} = 1 + \frac{11.3 \cdot \theta_3}{e^{2u}},$$

with $|\theta_3| < 1$. This result inserted into (14) yields

$$\left|\frac{d^2u}{dt^2}\right| < \frac{e^u \pm e^{-u}}{e^u \mp e^{-u}} \cdot \frac{22.65}{e^{2u}} < \frac{22.7}{e^{2u}}. \tag{16}$$

Using now twice the mean value theorem we obtain by (16)

$$\left|\frac{\frac{u_3-u_2}{t_3-t_2} - \frac{u_2-u_1}{t_2-t_1}}{t_3 - t_1}\right| < \frac{22.7}{e^{2u_1}}$$

and therefore we obtain by recalling the definition of the t's and u's

$$\left|\frac{k_3 - k_2}{j_3 - j_2} - \frac{k_2 - k_1}{j_2 - j_1}\right| < \frac{22.7(\log\beta)^2(j_3 - j_1)}{e^{2u_1}\log\alpha}.$$

Similarly as in [5, pp. 410-1] we conclude

$$j_3 - j_2 > \frac{j_1 k_1 \alpha^{2k_1} \log \alpha}{22.7(j_3 - j_1)(j_2 - j_1)(\log \beta)^2}. \tag{17}$$

Now by Proposition 2 we have $j_3 - j_1 < 312 j_1$ and also

$$j_1 \log(\beta) < \log(\alpha^{k_1}) + \log(32) + 1 < 1.34 \log(\alpha^{k_1}),$$

hence we obtain

$$j_3 - j_2 > j_1 k_1 \frac{\alpha^{2k_1}}{22.7 \cdot 312^2 \cdot 1.34^2 k_1^2 \log \alpha} > 337 j_1 k_1. \quad \square$$

As an immediate consequence of Lemmas 5 and 6 in combination with Proposition 2, we obtain Proposition 3.

REMARK 1. Since we assume that the smallest solution to (8) is large, i.e. $k_1 \geqq 28$, our proof that there exist only 2 solutions avoids the main problem of proving that there are at most two solutions. Therefore the proof of Proposition 3 is rather short, compared to that of the result due to Cipu and Mignotte [7].

6. Proof of Theorem 1

First, let us note an almost immediate consequence of Lemma 2:

LEMMA 7. *Assume that* $k \geqq 28$ *in the notation of Theorem* 1. *Then we have* $z > 16m^4$.

PROOF. The estimate (6) shows that if ε is the fundamental solution to the first equation of (3), then we have

$$z > \frac{m^{13.5}}{2^{28}} > 16m^4,$$

provided $m > 10.4$.

Assume that ε is the fundamental solution to the second equation of (3). Then we get by Lemma 2

$$z\sqrt{2^{-\delta}d} + x\sqrt{2^{\delta}n} = \left(z_0\sqrt{2^{-\delta}d} + x_0\sqrt{2^{\delta}n}\right)^k$$

with some k. A trivial estimate together with the remark at the end of Section 2 leads to the inequality

$$z > \frac{\max\left\{n, 2^{-\delta}d\right\}^{13.5}}{2} > \frac{(m/8)^{13.5}}{2} > 16m^4,$$

provided $m > 24.8$.

In the case that ε is the fundamental solution to the third equation, note that $x, y > 16m^4$ provided $m > 1.45$. Considering now the third and second equation of (3) we deduce $z > y > 16m^4$. On the other hand $m \geqq 3$ and therefore this equation is always fulfilled.

Hence we are left with finitely many cases. For each case we can compute the fundamental solution and therefore also a lower bound for z under the assumption $k \geqq 28$. Indeed we obtain $z > 16m^4$. $\square$

Assume now that there exists a UPIB, in particular assume $\mathbb{Z}_{\mathbb{K}} = \mathbb{Z}[\theta]$ with $\theta \in \mathbb{Z}_{\mathbb{K}}^*$ and let (x, y, z) and (x_i, y_i, z_i) be the solutions to (3) that correspond to θ and θ^{-1}, respectively. Without loss of generality we may assume that the exponents k and k_i corresponding to θ and θ^{-1} satisfy $k \leqq k_i$ and by Proposition 3 we have $k|k_i$. But this implies $z|z_i$. On the other hand θ^{-1} is explicitly computable in terms of a, x, y, z. In particular, for our purpose the quantity z_i is interesting and we obtain

$$\pm 4z_i = -4a^2 z - 4\delta mnx^2 z - ny\big(dzy + 2^{1+\delta}x^2(\delta - 1)\big)$$

$$+ 4ax\big(2^{\delta}ny + z(\delta - 1)\big) + z\big(dmz^2 - x^2(\delta - 1)^2\big).$$

In the following we will distinguish between the two cases $\delta = 1$ and $\delta = 0$.

6.1. *The case* $\delta = 1$. First, note that in this case $m = n + d$ and we have

$$0 \equiv z_i \equiv \pm 2anxy \bmod z.$$

We have excluded the case $x = 0$ or $y = 0$ by assuming positive solutions. Also neither x nor y has a common divisor with z since otherwise the second or the first Pell equation in (3) cannot be fulfilled. Also z and n can only have a common divisor dividing 2. But z has to be odd, since otherwise both y and z have to be even, which contradicts the first equation of (3). Therefore we conclude $z|a$.

Now let us compute the norm of the unit θ with respect to the assumption $\delta = 1$. We obtain

$$\pm 1 = \mathrm{N}(\theta) = \frac{1}{16} \times \big(16a^4 + d^2n^2y^4 + 32admnxyz + m^2\big(-4nx^2 + dz^2\big)^2$$

$$- 2dmny^2\big(4nx^2 + dz^2\big) - 8a^2\big(4mnx^2 + dny^2 + dmz^2\big)\big).$$

Let us consider the equation for the norm modulo z. Then we obtain with $z|a$ in mind

$$\pm 1 \equiv \frac{n^2\big(dy^2 - 4mx^2\big)^2}{16} \bmod z.$$

Expressing x^2 and y^2 in terms of z^2 by the second and third equation of (3) yields

$$\pm 4 \equiv (d-m)^2 \bmod z.$$

Since $m \geqq 3$ and Lemma 7 we have

$$(d-m)^2 < m^2 < 8m^4 < \frac{z}{2}$$

and therefore $(d-m)^2 = \pm 4$. This is a contradiction unless $m - d = n = 2$. However we assumed n to be odd, thus $n = 2$ is impossible.

6.2. *The case* $\delta = 0$. In this case $m = n + 4d$ and we deduce

$$0 \equiv 2z_i \equiv nx(2a+x)y \bmod z.$$

Since eventually z and y have greatest common divisor 2 and since z is relatively prime to nx we obtain $(z/2) \mid 2a + x$ if z is even and $z \mid 2a + x$ otherwise. Again we compute the norm of θ modulo z and obtain

$$\pm 1 \equiv \mathrm{N}(\theta) = \frac{n^2\left(mx^2 - dy^2\right)^2}{16} \bmod z.$$

Now we use the second and the third equation of (3) to express y^2 and x^2 in terms of z^2 and obtain

$$\pm 1 \equiv \frac{(m-4d)^2}{16} \equiv \frac{n^2}{16} \bmod z.$$

Assuming $z = 2^{p_2} z_1$ with z_1 odd we obtain, by multiplying the equation above by 16,

$$\pm 16 \equiv n^2 \bmod \frac{z}{2^{\min\{p_2,4\}}}.$$

Since $m \geqq 3$ and Lemma 7 we have $n^2 < m^2 < z/16$ and the plus sign cannot hold, unless $n = 4$ which contradicts the assumption that n is square-free. On the other hand a square root of -16 modulo $\frac{z}{2^{\min\{p_2,4\}}}$ is at least

$$\sqrt{\frac{z}{16} - 16} > \sqrt{m^4 - 16} > m^2 - \frac{8}{m^2} > m^2 - 1 \geqq n^2$$

which also yields a contradiction.

Hence in both cases we have shown that the solutions corresponding to θ and θ^{-1} cannot be related by $k|k_i$ and therefore Proposition 3 immediately yields Theorem 1.

7. Trivial biquadratic fields

By Definition 1 the biquadratic field $\mathbb{Q}(\sqrt{dm}, \sqrt{dn})$ given in canonic form is trivial, if either $d = 2^\delta$ or $|m| = 1$ or $|n| = 1$. According to Gras and Tanoé [15, Theorem 13] we know that the only trivial real biquadratic fields, whose maximal orders have PIBs, are contained in the families

• $\mathbb{K} = \mathbb{Q}\Big(\sqrt{\frac{n-1}{4}}, \sqrt{n}\Big)$, i.e. $d = \frac{n-1}{4}$, $m = 1$, $\delta = 0$ and $n \equiv 9, 13 \bmod 16$, n and $(n-1)/4$ are square-free;

• $\mathbb{K} = \mathbb{Q}\big(\sqrt{n+4}, \sqrt{n}\big)$, i.e. $d = 1$, $m = n+4$, $\delta = 0$ and $n \equiv 3 \bmod 4$, n and $n+4$ are square-free;

• $\mathbb{K} = \mathbb{Q}\Big(\sqrt{2(n+2)}, \sqrt{2n}\Big)$, i.e. $d = 2$, $m = n+2$, $\delta = 1$ and n odd, n and $n+2$ are square-free.

Let us treat the last two cases first. In this case trivial solutions to (3) exist. Indeed $(x, y, z) = (0, 1, 1)$ is in both cases a solution and we obtain $\mathbb{Z}_\mathbb{K} = \mathbb{Z}[\theta]$ with

$$\theta = \frac{\sqrt{dn} + \sqrt{dm}}{2} = \begin{cases} \dfrac{\sqrt{n} + \sqrt{n+4}}{2} & \text{if } \ d = 1, \ m = n+4; \\ \dfrac{\sqrt{2n} + \sqrt{2n+4}}{2} & \text{if } \ d = 2, \ m = n+2. \end{cases}$$

Obviously in both cases θ is a unit and therefore a UPIB exists.

Now let us consider the first case, i.e. we have to consider the system

$$nz^2 - y^2 = -4, \quad \frac{n-1}{4}z^2 - x^2 = -1, \quad \frac{n-1}{4}y^2 - nx^2 = -1. \tag{18}$$

The last equation has fundamental solution $\varepsilon = 2\sqrt{(n-1)/4} + \sqrt{n}$. Therefore the system of Pell equations with opposite signs has no solution due to Lemma 4. According to Theorem 1 we have to check whether there exist solutions to (18) coming from ε^k for $1 \leqq k \leqq 27$. By Lemma 2 we know that k has to be odd.

First let us assume $k > 1$. Then it is easy to compute z^2 for the possible 13 cases. Computing z^2 modulo 256 we see that in most cases, i.e. $k \neq 7, 9, 23, 25$, we have either $32 \parallel z^2$ (if $k = 3, 5, 11, 13, 19, 21, 27$) or $128 \parallel z^2$ (if $k = 15, 17$), which is impossible. Hence we are left with the cases $k = 7, 9, 23, 25$.

Now, we obtain the following expressions for z^2.

$$k = 7 : z^2 = 64(2n-1)(4n-3)(4n-1)(8n^2 - 8n + 1)$$

$$k = 9 : z^2 = 64(2n-1)(8n^2 - 8n + 1)(16n^2 - 20n + 5)(16n^2 - 12n + 1)$$

$$k = 23 : z^2 = 64(2n-1)(4n-3)(4n-1)(8n^2 - 8n + 1)$$

$$\times (16n^2-16n+1)(256n^4-512n^3+320n^2-64n+1)$$

$$\times (1024n^5-2816n^4+2816n^3-1232n^2+220n-11)$$

$$\times (1024n^5-2304n^4+1792n^3-560n^2+60n-1)$$

$$k=25: z^2=64(2n-1)(4n-3)(4n-1)(8n^2-8n+1)$$

$$\times (16n^2-16n+1)(256n^4-512n^3+320n^2-64n+1)$$

$$\times (4096n^6-13312n^5+16640n^4-9984n^3+2912n^2-364n+13)$$

$$\times (4096n^6-11264n^5+11520n^4-5376n^3+1120n^2-84n+1).$$

It is easy to show that in the cases $k=7,23$ and 25 the factor $4n-1$ is prime to the other factors, hence $4n-1$ has to be square, a contradiction. In the case of $k=9$ we also see that the factors are coprime, hence $2n-1=\xi^2$ and $8n^2-8n+1=\eta^2$. Thus we obtain the Diophantine equation

$$2\xi^4-\eta^2=1.$$

This equation has been already considered by Ljunggren [19] (see also [6]) and the only solutions are $(1,1)$ and $(13,239)$. But $\xi=1$ yields $n=1$ which is not an admissible value. The case $\xi=13$ yields $n=85$, but this value does not yield an integer for z.

Now let us consider the case $k=1$. In this case there exists the non-negative solution $(x,y,z)=(1,2,0)$. Hence Theorem 1 is still applicable. We are left to check whether

$$\theta=a+\frac{1+\sqrt{n}}{2}\pm\sqrt{\frac{n-1}{4}}$$

is a unit for some $a\in\mathbb{Z}$. Computing the norm of θ we obtain:

$$\left(1+2a+2a^2\right)^2-n(1+2a)^2=\pm 4.$$

This is a linear equation in n and we obtain

$$n=a^2+a+\frac{3}{4}+\frac{1\pm 16}{4(2a+1)^2}.$$

Since

$$\left|\frac{1\pm 16}{4(2a+1)^2}\right|\leqq\frac{1}{4}$$

unless $-3 \leqq a \leqq 2$, we deduce that n cannot be an integer for "large" a. We compute n for each of the remaining a's in both cases and obtain only the integer solutions $n = -3, 5$. Since we assume $n > 0$ and $n \equiv 9, 13 \bmod 16$, these solutions do not yield fields with UPIBs.

Therefore we have proved Theorem 2 and also Corollary 1.

8. The family $\mathbb{Q}(\sqrt{18n^2+17n+4}, \sqrt{2n^2+n})$

In this section we want to show what will be the typical result if Theorem 1 is applied to families of biquadratic fields. In particular, we choose $m = 9n + 4$ and $d = 2n + 1$. The aim of this section is to prove

THEOREM 4. *Assume that n is odd and $18n^2+17n+4$ and $2n^2+n$ are square-free. Then $\mathbb{K} = \mathbb{Q}(\sqrt{18n^2+17n+4}, \sqrt{2n^2+n})$ is a quartic field and there exists an explicit computable constant C such that for $n > C$ the field $\mathbb{K}$ admits no UPIB.*

REMARK 2. It is not obvious that $18n^2 + 17n + 4$ and $2n^2 + n$ are simultaneously square-free for infinitely many n. Erdős [8, pp. 417–418] points out that $f(n) = h(n)g(n)$ is square-free for infinitely many n, provided the polynomials $g(n)$ and $h(n)$ are coprime and both are square-free for infinitely many n. Applied to $f(n) = (18n^2 + 17n + 4)(2n^2 + n)$ we see that indeed, for infinitely many n, both $18n^2 + 17n + 4$ and $2n^2 + n$ are square-free simultaneously.

PROOF. In view of Theorem 1 we have to consider the system of Pell equations

$$\begin{cases} (9n+4)z^2 - y^2 n = \pm 4, \\ (2n+1)z^2 - nx^2 = \pm 1, \\ (2n+1)y^2 - (9n+4)x^2 = \pm 1. \end{cases} \tag{19}$$

The first equation has the fundamental solution $\varepsilon = \sqrt{9n+4} - 3\sqrt{n}$ and therefore the "+"-signs hold in (19). Hence the "−"-sign can be excluded due to Lemma 4. Thus we have to consider potential solutions induced by ε^k for $1 \leqq k \leqq 27$ and k odd. We compute for each k the value of y^2. For the first k's we obtain

$$k = 1: \ y^2 = 2$$

$$k = 3: \ y^2 = (9n+4)(18n+5)$$

$$k = 5: \ y^2 = (81n^2+54n+7)(162n^2+81n+8)$$

and for $7 \leqq k \leqq 27$ we get

$$y^2 = \left(3^{k-1}n^{(k-1)/2} + \cdots\right)\left(2 \cdot 3^{k-1}n^{(k-1)/2} + \cdots\right) = p_k(n)q_k(n),$$

where the factors are irreducible polynomials which take coprime values for all n. This can be seen by computing the extended polynomial gcd

$$a_k(n)p_k(n) + b_k(n)q_k(n) = 1$$

over $\mathbb{Q}[n]$ and observing that $a_k(n), b_k(n) \in \frac{1}{3}\mathbb{Z}[n]$, but $3 \mid a_k(n)$ nor $3 \mid b_k(n)$ for any n. Hence for each $k > 5$ we have a hyperelliptic equation which can be solved in theory explicitly (cf. [2]). In practice the degree and the coefficients of $p_k(n)$ are too large to solve the corresponding hyperelliptic equations explicitly. However, there are only finitely many n which yield solutions for any of these Diophantine equations.

So we are left to the cases $k = 1, 3, 5$. The case $k = 1$ obviously yields no solution. In the case $k = 3$ we write $\xi^2 = 9n + 4$ and $\eta^2 = 18n + 5$ which leads to the Pell equation

$$\eta^2 - 2\xi^2 = -3,$$

which has no solution. Indeed consider the equation modulo 3, then in case of a solution we would have $\left(\frac{2}{3}\right) = 1$ for the Legendre symbol, a contradiction. Now let us consider the case $k = 5$. Since the factors $81n^2 + 54n + 7$ and $162n^2 + 81n + 8$ are coprime, the first factor has to be a square. But,

$$81n^2 + 54n + 7 = (9n + 3)^2 - 2 = \square$$

is again a contradiction, which finally proves Theorem 4. $\square$

References

[1] N. Ashrafi and P. Vámos, On the unit sum number of some rings, *Quarterly J. Math.*, **56** (2005), 1–12.

[2] A. Baker, Bounds for the solutions of the hyperelliptic equation, *Proc. Cambridge Philos. Soc.*, **65** (1969), 439–444.

[3] P. Belcher, Integers expressible as sums of distinct units, *Bull. Lond. Math. Soc.*, **6** (1974), 66–68.

[4] M. A. Bennett, On the number of solutions of simultaneous Pell equations, *J. Reine Angew. Math.*, **498** (1998), 173–199.

[5] M. A. Bennett, M. Cipu, M. Mignotte and R. Okazaki, On the number of solutions of simultaneous Pell equations. II, *Acta Arith.*, **122** (2006), 407–417.

[6] J. H. Chen and P. Voutier, Complete solution of the Diophantine equation $X^2 + 1 = dY^4$ and a related family of quartic Thue equations, *J. Number Theory*, **62** (1997), 71–99.

[7] M. Cipu and M. Mignotte, On the number of solutions to systems of Pell equations, *J. Number Theory*, **125** (2007), 356–392.

[8] P. Erdős, Arithmetical properties of polynomials, *J. London Math. Soc.*, **28** (1953), 416–425.
[9] A. Filipin, R. Tichy and V. Ziegler, The additive unit structure of pure quartic complex fields, *Funct. Approx. Comment. Math.*, **39** (2008), 113–131.
[10] I. Gaál, Solving index form equations in fields of degree 9 with cubic subfields, *J. Symbolic Comput.*, **30** (2000), 181–193.
[11] I. Gaál, *Diophantine Equations and Power Integral Bases*, Birkhäuser Boston Inc. (Boston, MA, 2002).
[12] I. Gaál, P. Olajos and M. Pohst, Power integral bases in orders of composite fields, *Experiment. Math.*, **11** (2002), 87–90.
[13] I. Gaál, A. Pethő and M. Pohst, On the resolution of index form equations in biquadratic number fields. III. The bicyclic biquadratic case, *J. Number Theory*, **53** (1995), 100–114.
[14] I. Gaál and N. Schulte, Computing all power integral bases of cubic fields, *Math. Comp.*, **53** (1989), 689–696.
[15] M.-N. Gras and F. Tanoé, Corps biquadratiques monogènes, *Manuscripta Math.*, **86** (1995), 63–79.
[16] K. Győry, Bounds for the solutions of norm form, discirminant form and index form equations in finitely generated integral domains, *Acta Math. Hungar.*, **42** (1983), 45–80.
[17] K. Győry, Sur les polynômes à coefficients entiers et de discriminant donné. III, *Publ. Math. Debrecen*, **23** (1976), 141–165.
[18] M. Jarden and W. Narkiewicz, On sums of units, *Monatsh. Math.*, **150** (2007), 327–336.
[19] W. Ljunggren, Zur Theorie der Gleichung $x^2 + 1 = Dy^4$, *Avh. Norske Vid. Akad. Oslo. I.*, **27** (1942).
[20] T. Nagell, On a special class of Diophantine equations of the second degree, *Ark. Mat.*, **3** (1954), 51–65.
[21] R. Tichy and V. Ziegler, Units generating the ring of integers of complex cubic fields, *Coll. Math.*, **109** (2007), 71–83.
[22] K. S. Williams, Integers of biquadratic fields, *Canad. Math. Bull.*, **13** (1970), 519–526.
[23] V. Ziegler, The additive S-unit structure of quadratic fields, to appear in *Int. J. Number Theory*.
[24] V. Ziegler, The additive unit structure of complex biquadratic fields, *Glas. Mat. Ser. III*, **43** (2008), 293–307.