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Research Article

q-Hyperconvexity in Quasipseudometric Spaces and Fixed Point Theorems

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In a previous work, we started investigating the concept of hyperconvexity in quasipseudometric spaces which we called q-hyperconvexity or Isbell-convexity. In this paper, we continue our studies of this concept, generalizing further known results about hyperconvexity from the metric setting to our theory. In particular, in the present paper, we consider subspaces of q-hyperconvex spaces and also present some fixed point theorems for nonexpansive self-maps on a bounded q-hyperconvex quasipseudometric space. In analogy with a metric result, we show among other things that a set-valued mapping T^* on a q-hyperconvex T_0-quasimetric space (X, d) which takes values in the space of nonempty externally q-hyperconvex subsets of (X, d) always has a single-valued selection T which satisfies $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ whenever $x, y \in X$. (Here, d_H denotes the usual (extended) Hausdorff quasipseudometric determined by d on the set $\mathcal{P}_0(X)$ of nonempty subsets of X.)

1. Introduction

In a previous work, we started investigating a concept of hyperconvexity in quasipseudometric spaces, which we called q-hyperconvexity or Isbell-convexity (see [1], compare [2]). In this paper, we continue our studies of this concept by generalizing further known results about hyperconvexity from the metric setting to our theory. Among other things, in the present paper we consider subspaces of q-hyperconvex spaces and also present some fixed point theorems. In particular, we show that a set-valued mapping T^* on a q-hyperconvex T_0-quasimetric space (X, d) which takes values in the space of nonempty externally q-hyperconvex subsets of (X, d) always has a single-valued selection T which satisfies $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ whenever $x, y \in X$. (Here, d_H denotes the usual (extended) Hausdorff quasipseudometric determined by d on the set $\mathcal{P}_0(X)$ of nonempty subsets of X.)

Our investigations confirm the surprising fact that many classical results about hyperconvexity in metric spaces do not make essential use of the symmetry of the metric and, therefore, still hold—in a sometimes slightly modified form—for our concept of q-hyperconvexity in quasipseudometric spaces (see also [3] for a more general approach).

For the basic facts concerning quasipseudometrics and quasiuniformities we refer the reader to [4, 5]. Some recent work about quasipseudometric spaces can be found in the articles [6–9].

2. Preliminaries

In order to fix the terminology, we start with some basic concepts.

Definition 2.1. Let X be a set and let $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then, d is called a quasipseudometric on X if

(a) $d(x, x) = 0$ whenever $x \in X$,

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We will say that d is a T_0-*quasimetric* provided that d also satisfies the following condition: for each $x, y \in X$,

$$d(x, y) = 0 = d(y, x) \text{ implies that } x = y. \tag{2.1}$$

Remark 2.2. Let d be a quasipseudometric on a set X, then $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasipseudometric, called the conjugate quasipseudometric of d. As usual, a quasipseudometric d on X such that $d = d^{-1}$ is called a pseudometric. Note that for any (T_0)-quasipseudometric d, the function $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a pseudometric (metric).

For any $a, b \in [0, \infty)$, we will set $a \dot{-} b = \max\{a - b, 0\}$.

Let (X, d) be a quasipseudometric space. For each $x \in X$ and $\epsilon > 0$, the set $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ denotes the open ϵ-*ball* at x. The collection of all "open" balls yields a base for a topology $\tau(d)$. It is called the *topology induced by* d on X. Similarly, for each $x \in X$ and $\epsilon \geq 0$, we define the ball $C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$. Note that this latter set is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general. As usual, in the theory of quasiuniformities, for a subset A of X and $\epsilon > 0$, we will also use notations like $B_d(A, \epsilon) = \bigcup_{a \in A} B_d(a, \epsilon)$ and similarly $C_d(A, \epsilon) = \bigcup_{a \in A} C_d(a, \epsilon)$.

A pair $(C_d(x, r); C_{d^{-1}}(x, s))$ with $x \in X$ and nonnegative reals r, s will be called a *double ball at* x.

We shall also speak of a family $[(C_d(x_i, r_i))_{i \in I}; (C_{d^{-1}}(x_i, s_i))_{i \in I}]$ of double balls, with $x_i \in X$ and $r_i, s_i \geq 0$ whenever $i \in I$.

Let (X, d) be a quasipseudometric space and let $\mathcal{P}_0(X)$ be the set of all nonempty subsets of X. Given $C \in \mathcal{P}_0(X)$, we will set dist $(x, C) = \inf\{d(x, c) : c \in C\}$ and dist $(C, x) = \inf\{d(c, x) : c \in C\}$ whenever $x \in X$.

For any $A, B \in \mathcal{P}_0(X)$, we set

$$d_H(A,B) = \max\left\{ \sup_{b\in B} \operatorname{dist}(A,b), \sup_{a\in A} \operatorname{dist}(a,B) \right\} \tag{2.2}$$

(compare [10]).

Then d_H, is the so-called extended (as usual, a quasipseudometric that maps into $[0,\infty]$ (instead of $[0,\infty)$) will be called *extended*) *Hausdorff(-Bourbaki) quasipseudometric* on $\mathcal{P}_0(X)$. It is known that d_H is an extended T_0-quasimetric when restricted to the set of all the nonempty subsets A of X which satisfy $A = \mathrm{cl}_{\tau(d)} A \cap \mathrm{cl}_{\tau(d^{-1})} A$ (compare [11, page 164]).

A map $f : (X,d) \to (Y,e)$ between two quasipseudometric spaces (X,d) and (Y,e) is called an *isometry* or *isometric map* provided that $e(f(x), f(y)) = d(x,y)$ whenever $x, y \in X$. Two quasipseudometric spaces (X,d) and (Y,e) will be called *isometric* provided that there exists a bijective isometry $f : (X,d) \to (Y,e)$. A map $f : (X,d) \to (Y,e)$ between two quasipseudometric spaces (X,d) and (Y,e) is called *nonexpansive* provided that $e(f(x), f(y)) \le d(x,y)$ whenever $x, y \in X$.

The following definitions can be found in [1] (compare [12]).

Definition 2.3 (see [1, Definition 2]). A quasipseudometric space (X,d) is called q-hyperconvex (or Isbell-convex) provided that for each family $(x_i)_{i\in I}$ of points in X and families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of nonnegative real numbers satisfying $d(x_i, x_j) \le r_i + s_j$ whenever $i, j \in I$, the following condition holds:

$$\bigcap_{i\in I} \left(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \right) \neq \emptyset. \tag{2.3}$$

Definition 2.4 (see [1, Definition 5]). Let (X,d) be a quasipseudometric space. A family of double balls $[(C_d(x_i, r_i))_{i\in I}; (C_{d^{-1}}(x_i, s_i))_{i\in I}]$ with $r_i, s_i \in [0,\infty)$ and $x_i \in X$ whenever $i \in I$ is said to have the mixed binary intersection property if for all indices $i, j \in I, C_d(x_i, r_i) \cap C_{d^{-1}}(x_j, s_j) \neq \emptyset$.

Definition 2.5 (see [1, Definition 6]). A quasipseudometric space (X,d) is called q-hypercomplete (or Isbell-complete) if every family

$$[(C_d(x_i, r_i))_{i\in I}; (C_{d^{-1}}(x_i, s_i))_{i\in I}] \tag{2.4}$$

of double balls, where $r_i, s_i \ge 0$ and $x_i \in X$ whenever $i \in I$, having the mixed binary intersection property satisfies $\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$.

Definition 2.6 (see [1, Definition 4]). Let (X,d) be a quasipseudometric space. We say that X is metrically convex if for any points $x, y \in X$ and nonnegative real numbers r and s such that $d(x,y) \le r + s$, there exists $z \in X$ such that $d(x,z) \le r$ and $d(z,y) \le s$.

The following useful result was established in [1, Proposition 1]. A quasipseudometric space (X,d) is q-hyperconvex if and only if it is metrically convex and q-hypercomplete.

As usual, a subset A of a quasipseudometric space (X,d) will be called *bounded* provided that there is a positive real constant M such that $d(x,y) < M$ whenever $x, y \in A$.

Note that a subset A of (X,d) is bounded if and only if there are $x \in X$ and $r,s \geq 0$ such that $A \subseteq C_d(x,r) \cap C_{d^{-1}}(x,s)$.

3. Some First Results

Proposition 3.1 (compare [13, Proposition 4.5]). *Let (X,d) be a q-hyperconvex quasipseudometric space. Let $(x_i)_{i\in I}$ be a nonempty family of points in X and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative reals such that $d(x_i,x_j) \leq r_i + s_j$. Set $D = \bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i))$. Then D is nonempty and q-hyperconvex.*

Proof. Note first that $D \neq \emptyset$ by q-hyperconvexity of X. For each $\alpha \in S$, let $x_\alpha \in D$ and let r_α, s_α be nonnegative reals such that $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$ whenever $\alpha, \beta \in S$.

We show that the family

$$[(C_d(x_\alpha,r_\alpha))_{\alpha\in S}, (C_d(x_i,r_i))_{i\in I}; (C_{d^{-1}}(x_\alpha,s_\alpha))_{\alpha\in S}, (C_{d^{-1}}(x_i,s_i))_{i\in I}] \tag{3.1}$$

satisfies the hypothesis of q-hyperconvexity. Indeed, in particular, for each $\alpha \in S$ and $i \in I$, we have that $d(x_\alpha,x_i) \leq s_i \leq r_\alpha + s_i$ and $d(x_i,x_\alpha) \leq r_i \leq r_i + s_\alpha$.

Hence, by q-hyperconvexity of X, we have that

$$\begin{aligned} \emptyset &\neq \bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)) \cap \bigcap_{\alpha\in S}(C_d(x_\alpha,r_\alpha) \cap C_{d^{-1}}(x_\alpha,s_\alpha)) \\ &= D \cap \bigcap_{\alpha\in S}(C_d(x_\alpha,r_\alpha) \cap C_{d^{-1}}(x_\alpha,s_\alpha)). \end{aligned} \tag{3.2}$$

Hence, the subspace D of X is q-hyperconvex. □

Let (X,d) be a quasipseudometric space. For a nonempty bounded subset A of X, we set

$$\begin{aligned} \text{bicov }(A)_+ &= \bigcap\{C_d(x,r) : A \subseteq C_d(x,r),\ x \in X,\ r \geq 0\}, \\ \text{bicov }(A)_- &= \bigcap\{C_{d^{-1}}(x,s) : A \subseteq C_{d^{-1}}(x,s),\ x \in X,\ s \geq 0\}. \end{aligned} \tag{3.3}$$

Furthermore, we define the *bicover of* A by bicov $(A) :=$ bicov $(A)_+ \cap$ bicov $(A)_-$.

A nonempty bounded set A in a quasipseudometric space (X,d) that can be written as the intersection of a nonempty family of sets of the form $C_d(x,\epsilon_1) \cap C_{d^{-1}}(x,\epsilon_2)$ where $\epsilon_1,\epsilon_2 \geq 0$ and $x \in X$, that is, $A =$ bicov A, will be called *q-admissible* in the following. By $\mathcal{A}_q(X)$, we will denote the *set of q-admissible subsets of* X. Note that if (X,d) is q-hyperconvex, then any member of $\mathcal{A}_q(X)$ is q-hyperconvex in the light of Proposition 3.1.

Let (X,d) be a quasipseudometric space and let A be a nonempty bounded subset in (X,d). Then, in accordance with [13, page 79], we can define *the cover* cov A *of* A as follows: cov $A = \bigcap\{C_{d^s}(x) : A \subseteq C_{d^s}(x), x \in X\}$. Obviously, we have $A \subseteq$ bicov $(A) \subseteq$ cov (A). The latter inclusion can be strict, as our first example shows.

Example 3.2. Let $X = [0,1] \times [1/4,3/4]$ be equipped with the T_0-quasimetric d defined by $d((\alpha,\beta),(\alpha',\beta')) = (\alpha \dot{-} \alpha') \vee (\beta \dot{-} \beta')$ whenever $(\alpha,\beta),(\alpha',\beta') \in X$.

Consider $A = \{(0,1/2),(1,1/2)\} \subseteq X$. Then, bicov (A) is equal to the line segment in X from $x = (0,1/2)$ to $y = (1,1/2)$. This follows from the fact that, for each $\epsilon \in [0,1/4]$, we have $x \in [0,1] \times [1/4,(1/2)+\epsilon] = C_{d^{-1}}(y,\epsilon)$ and $y \in [0,1] \times [(1/2)-\epsilon,3/4] = C_d(x,\epsilon)$, and that the line segment is a subset of any set of the form $C_d(a,r) \cap C_{d^{-1}}(b,s)$ for which $\{x,y\} \subseteq C_d(a,r) \cap C_{d^{-1}}(b,s)$. Indeed, assume that the point z belongs to this segment. Then $d(z,y) = 0 = d(x,z)$ and, therefore, $z \in C_d(a,r) \cap C_{d^{-1}}(b,s)$ by the triangle inequality.

On the other hand, $\operatorname{cov}(A) = X$, since $\{x,y\} \subseteq C_{d^s}(z,\epsilon)$ with $z \in X$ implies that $\epsilon \geq 1/2$. Indeed, assume that $z = (a,b) \in X$. Then, $a \leq d^s((a,b),(0,1/2)) \leq \epsilon$ and $1-a \leq d^s((a,b),(1,1/2)) \leq \epsilon$. Thus, $\epsilon \geq \max\{a,1-a\} \geq 1/2$ with $a \in [0,1]$. In the light that the interval $[1/4,3/4]$ has length $1/2$, it follows that $X \subseteq C_{d^s}(z,\epsilon)$. Therefore, $\operatorname{cov}(A) = X$.

By the results of [1, Example 1], $(\operatorname{bicov}(A),d)$ is q-hyperconvex, while the metric space $(\operatorname{bicov}(A),d^s)$ is hyperconvex [1, Proposition 2], but not q-hyperconvex (see [1, Example 2]).

The following result gives a quasipseudometric variant of a well-known result usually attibuted to Sine [14] (compare also [15]).

Theorem 3.3. *If (X,d) is a bounded q-hyperconvex T_0-quasimetric space and if $T : (X,d) \to (X,d)$ is a nonexpansive map, then the fixed point set $\operatorname{Fix}(T)$ of T in (X,d) is nonempty and q-hyperconvex.*

Proof. We first show that $\operatorname{Fix}(T) \neq \emptyset$. Note that $T : (X,d^s) \to (X,d^s)$ is nonexpansive, since for any $x,y \in X$, we have $d(Tx,Ty) \leq d(x,y)$ and $d(Ty,Tx) \leq d(y,x)$, and thus $d^s(Tx,Ty) \leq d^s(x,y)$. By assumption (X,d^s) is bounded. Furthermore, (X,d^s) is a hyperconvex space according to [1, Proposition 2]. So, by [13, Theorem 4.8], we know that T has a fixed point and $\operatorname{Fix}(T)$ is hyperconvex in (X,d^s).

We need to show that $\operatorname{Fix}(T)$ is indeed q-hyperconvex. Let

$$[(C_d(x_i,r_i))_{i\in I};(C_{d^{-1}}(x_i,s_i))_{i\in I}] \tag{3.4}$$

be a nonempty family of double balls, where $x_i \in \operatorname{Fix}(T)$ and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ are two families of nonnegative reals such that $d(x_i,x_j) \leq r_i + s_j$ whenever $i,j \in I$. Since X is a q-hyperconvex T_0-quasimetric space, the set

$$X_0 = \bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)) \neq \emptyset. \tag{3.5}$$

Let $x \in X_0$. Then, $d(T(x),x_i) = d(T(x),T(x_i)) \leq d(x,x_i) \leq s_i$ and

$$d(x_i,T(x)) = d(T(x_i),T(x)) \leq d(x_i,x) \leq r_i \tag{3.6}$$

whenever $i \in I$. Thus, $T(x) \in X_0$ and we have $T : X_0 \to X_0$.

Moreover, X_0 is a bounded q-hyperconvex T_0-quasimetric space by Proposition 3.1. So the first part of the proof implies that T has a fixed point in X_0, which implies that $\operatorname{Fix(T)} \cap [\bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)] \neq \emptyset$. We have shown that $\operatorname{Fix}(T)$ is q-hyperconvex. □

4. Chains of q-Hyperconvex Subspaces

In this section, we will prove the analogue of a famous theorem due to Baillon [16].

Theorem 4.1. *Let (X, d) be a bounded T_0-quasimetric space and let $(H_i)_{i\in I}$ be a descending family of nonempty q-hyperconvex subsets of X, where one assumes that I is totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ hold if and only if $H_{i_2} \subseteq H_{i_1}$. Then, $\bigcap_{i\in I} H_i$ is nonempty and q-hyperconvex.*

Proof. We begin by showing that $\bigcap_{i\in I} H_i \neq \emptyset$. We first note that (X, d^s) is a bounded metric space and $(H_i)_{i\in I}$ is a descending chain of hyperconvex sets in (X, d^s) by [1, Proposition 2]. By the well-known result of Baillon [16, Theorem 7], we conclude that $\bigcap_{i\in I} H_i$ is nonempty and hyperconvex in (X, d^s).

In order to complete the proof, we need to show that $H = \bigcap_{i\in I} H_i$ is q-hyperconvex. Let a nonempty family $(x_\alpha)_{\alpha\in S}$ of points in H and families of nonnegative real numbers $(r_\alpha)_{\alpha\in S}$ and $(s_\alpha)_{\alpha\in S}$ be given such that $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$ whenever $\alpha, \beta \in S$. Fix $i \in I$. Since H_i is a q-hyperconvex space and since $x_\alpha \in H_i$ whenever $\alpha \in S$, therefore, $\mathfrak{D}_i = \bigcap_{\alpha\in S}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap H_i$ is nonempty and q-hyperconvex by the proof of Proposition 3.1 and thus a hyperconvex subset of (X, d^s) by [1, Proposition 2].

Thus by the first part of our present proof,

$$\begin{aligned}\emptyset \neq \bigcap_{i\in I} \mathfrak{D}_i &= \bigcap_{i\in I}\left[\bigcap_{\alpha\in S}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap H_i\right] \\ &= \bigcap_{\alpha\in S}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap \bigcap_{i\in I} H_i,\end{aligned} \tag{4.1}$$

since $(\mathfrak{D}_i)_{i\in I}$ is descending. This proves that $H = \bigcap_{i\in I} H_i$ is q-hyperconvex. □

Definition 4.2. Let (X, d) be a T_0-quasimetric space and let a family of nonexpansive maps $(T_i)_{i\in I}$, with $T_i : (X, d) \to (X, d)$, be given. We say that $(T_i)_{i\in I}$ is a commuting family if $T_i \circ T_j = T_j \circ T_i$ whenever $i, j \in I$.

Our next lemma is motivated by [16, Corollary 8].

Lemma 4.3. *If $(H_\alpha)_{\alpha\in S}$ is a family of bounded q-hyperconvex subsets of a T_0-quasimetric space X such that $\bigcap_{\alpha\in F} H_\alpha$ is nonempty and q-hyperconvex whenever $F \subseteq S$ is finite, then the intersection $\bigcap_{\alpha\in S} H_\alpha$ is nonempty and q-hyperconvex.*

Proof. Consider $\mathfrak{F} = \{I \subseteq S : \text{for all } J \text{ finite}, J \subseteq S, \ \bigcap_{I\cup J} H_\alpha \text{ is nonempty and } q\text{-hyperconvex}\}$.

Obviously $\emptyset \in \mathfrak{F}$ and $\mathfrak{F}$ satisfies the hypothesis of Zorn's lemma because of Theorem 4.1. Let I be maximal in $\mathfrak{F}$. Then, $I \cup \{\alpha\} \in \mathfrak{F}$ whenever $\alpha \in S$. Because of the maximality of I, we, therefore, have $\alpha \in I$ whenever $\alpha \in S$. □

The next result is a consequence of Theorems 3.3 and 4.1. It is analogous to [17, Theorem 6.2].

Theorem 4.4. *Let (X, d) be a bounded q-hyperconvex T_0-quasimetric space. Any commuting family of nonexpansive maps $(T_i)_{i\in I}$, with $T_i : (X, d) \to (X, d)$, has a common fixed point. Moreover, the common fixed point set*

$$\bigcap_{i\in I} \mathrm{Fix}(T_i), \tag{4.2}$$

is q-hyperconvex.

Proof. We observe that (X, d^s) is a bounded hyperconvex metric space by [1, Proposition 2], and for each $i \in I$, the map $T_i : (X, d^s) \to (X, d^s)$ is nonexpansive, as we noted before (see proof of Theorem 3.3). By Theorem 3.3, each T_i has a fixed point. Hence, there is $x \in X$ such that $T_i(x) = x$. We now show that, given any $j \in I$, we have that $T_j(\mathrm{Fix}(T_i)) \subseteq \mathrm{Fix}(T_i)$ Indeed, if for some $x \in X$, we have $x = T_i(x)$, then $T_j(x) = T_j(T_i(x)) = T_i(T_j(x))$. So $T_j(x) \in \mathrm{Fix}(T_i)$.

By Theorem 3.3, we conclude that $T_j : \mathrm{Fix}(T_i) \to \mathrm{Fix}(T_i)$ has a fixed point $y \in \mathrm{Fix}(T_i)$, which then is a fixed point of T_i and T_j. Therefore, the set of common fixed points of T_i and T_j is q-hyperconvex by Theorem 3.3. Hence, by induction for each finite family $(T_i)_{i \in F}$ of nonexpansive self-maps on X the set of common fixed points is nonempty and q-hyperconvex.

Since $\bigcap_{i \in F} \mathrm{Fix}(T_i)$ is nonempty subset and q-hyperconvex whenever F is a finite subset of I, by Lemma 4.3 we conclude that $\bigcap_{i \in I} \mathrm{Fix}(T_i)$ is nonempty and q-hyperconvex. □

5. Approximate Fixed Points

In this section, we investigate the approximation of fixed points by generalizing some results from [13] (compare [18]). We first define an ϵ_1, ϵ_2-parallel set of a subset in a quasipseudometric space similarly to [13, page 89].

Definition 5.1. Let (X, d) be a quasipseudometric space. Given a subset A of X, we define for $\epsilon_1, \epsilon_2 \geq 0$ the ϵ_1, ϵ_2-parallel set of A as

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcup_{a \in A} \left(C_d(a, \epsilon_2) \cap C_{d^{-1}}(a, \epsilon_1)\right). \tag{5.1}$$

(Note that for each $\epsilon > 0$ in particular $N_{\epsilon,\epsilon}(A) = \bigcup_{a \in A} C_{d^s}(a, \epsilon)$.)

Thus, $x \in N_{\epsilon_1,\epsilon_2}(A)$ if and only if there exists $a \in A$ such that $d(a, x) \leq \epsilon_2$ and $d^{-1}(a, x) \leq \epsilon_1$.

We next give a characterization of $N_{\epsilon_1,\epsilon_2}(A)$ if A is a q-admissible set in a q-hyperconvex quasipseudometric space (compare [13, Lemma 4.2]).

Lemma 5.2. *Let (X, d) be a q-hyperconvex quasipseudometric space and let A be a q-admissible subset of X, say $\emptyset \neq A = \bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ with $x_i \in X$ and r_i, s_i nonnegative reals whenever $i \in I \neq \emptyset$. Then, for each $\epsilon_1, \epsilon_2 \geq 0$,*

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcap_{i \in I} \left(C_d(x_i, r_i + \epsilon_2) \cap C_{d^{-1}}(x_i, s_i + \epsilon_1)\right). \tag{5.2}$$

Proof. Suppose that $y \in N_{\epsilon_1,\epsilon_2}(A)$. Then, $d(a, y) \leq \epsilon_2$ and $d(y, a) \leq \epsilon_1$ for some $a \in A$. But for each $i \in I$,

$$\begin{aligned} d(x_i, y) &\leq d(x_i, a) + d(a, y) \leq r_i + \epsilon_2, \\ d(y, x_i) &\leq d(y, a) + d(a, x_i) \leq \epsilon_1 + s_i. \end{aligned} \tag{5.3}$$

Then, for each $i \in I$, we have $y \in C_d(x_i, r_i + \epsilon_2)$ and $y \in C_{d^{-1}}(x_i, s_i + \epsilon_1)$. This proves that $N_{\epsilon_1,\epsilon_2}(A) \subseteq \bigcap_{i \in I} (C_d(x_i, r_i + \epsilon_2) \cap C_{d^{-1}}(x_i, s_i + \epsilon_1))$.

Now suppose that $y \in \bigcap_{i\in I}(C_d(x_i, r_i + \epsilon_2) \cap C_{d^{-1}}(x_i, s_i + \epsilon_1))$ and let $i \in I$.
Hence,

$$\begin{aligned} d(x_i, y) &\le r_i + \epsilon_2, \\ d(y, x_i) &\le \epsilon_1 + s_i. \end{aligned} \tag{5.4}$$

By definition of A and the triangle inequality, for any $a \in A$ and any $i, j \in I$ we must have that

$$d(x_i, x_j) \le d(x_i, a) + d(a, x_j) \le r_i + s_j. \tag{5.5}$$

Hence, $[(C_d(x_i, r_i))_{i\in I},\ C_d(y, \epsilon_1);\ (C_{d^{-1}}(x_i, s_i))_{i\in I}, C_{d^{-1}}(y, \epsilon_2)]$ satisfies the hypothesis in the definition of q-hyperconvexity of (X, d).

So, by q-hyperconvexity of X,

$$\begin{aligned} \emptyset \neq & \left(\bigcap_{i\in I} C_d(x_i, r_i)\right) \cap C_d(y, \epsilon_1) \cap \left(\bigcap_{i\in I} C_{d^{-1}}(x_i, s_i)\right) \cap C_{d^{-1}}(y, \epsilon_2) \\ &= \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap (C_d(y, \epsilon_1) \cap C_{d^{-1}}(y, \epsilon_2)) \\ &= A \cap (C_d(y, \epsilon_1) \cap C_{d^{-1}}(y, \epsilon_2)). \end{aligned} \tag{5.6}$$

Therefore, there is $a \in A$ such that $d(y, a) \le \epsilon_1$ and $d(a, y) \le \epsilon_2$. Hence, $y \in N_{\epsilon_1,\epsilon_2}(A)$ and the proof is complete. □

The following lemma will be needed in our discussion below of approximate fixed point sets.

Lemma 5.3 (compare [13, Lemma 4.3]). *Suppose that (X, d) is a q-hyperconvex T_0-quasimetric space and let A be a q-admissible subset of X. Then, for each $\epsilon_1, \epsilon_2 \ge 0$ there is a nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(A)$ onto A which has the property that $d(x, R(x)) \le \epsilon_1$ and $d(R(x), x) \le \epsilon_2$ whenever $x \in N_{\epsilon_1,\epsilon_2}(A)$.*

Proof. Assume $\emptyset \neq A = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ with $I \neq \emptyset$. By Lemma 5.2, we know that $N_{\epsilon_1,\epsilon_2}(A)$ is q-admissible in (X, d) and so $N_{\epsilon_1,\epsilon_2}(A)$ is itself q-hyperconvex by Proposition 3.1. Consider the family $\mathcal{F} = \{(D, R_D) : A \subseteq D \subseteq N_{\epsilon_1,\epsilon_2}(A)$ and $R_D : D \to A$ is a nonexpansive retraction such that $d(x, R(x)) \le \epsilon_1$ and $d(R(x), x) \le \epsilon_2$ whenever $x \in D\}$.

Let Id denote the identity map on A. Note that $(A, \mathrm{Id}) \in \mathcal{F}$. So $\mathcal{F} \neq \emptyset$. If one partially orders $\mathcal{F}$ in the usual way ($(D, R_D) \preccurlyeq (H, R_H)$ if and only if $D \subseteq H$ and R_H is an extension of R_D), then each chain in $(\mathcal{F}, \preccurlyeq)$ is bounded above. So by Zorn's lemma $\mathcal{F}$ has a maximal element which we denote by (D, R_D). We need to show that $D = N_{\epsilon_1,\epsilon_2}(A)$. Suppose that there exists $x \in N_{\epsilon_1,\epsilon_2}(A)$ such that $x \notin D$, and consider the set

$$\begin{aligned} C = &\left[\bigcap_{w\in D} (C_d(R_D(w), d(w,x)) \cap C_{d^{-1}}(R_D(w), d(x,w)))\right] \\ &\cap \left[\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))\right] \cap [C_d(x,\epsilon_1) \cap C_{d^{-1}}(x,\epsilon_2)]. \end{aligned} \tag{5.7}$$

First, we want to show that $C \neq \emptyset$, and in order to do this by [1, Proposition 1], we need only to show that the family

$$\begin{aligned} &[(C_d(R_D(w), d(w,x)))_{w\in D}, (C_d(x_i, r_i))_{i\in I}, C_d(x,\epsilon_1); \\ &\quad (C_{d^{-1}}(R_D(w), d(x,w)))_{w\in D}, (C_{d^{-1}}(x_i, s_i))_{i\in I}, C_{d^{-1}}(x,\epsilon_2)] \end{aligned} \tag{5.8}$$

of double balls has the mixed binary intersection property.

First note that if $w_1, w_2 \in D$, then

$$d(R_D(w_1), R_D(w_2)) \leq d(w_1, w_2) \leq d(w_1, x) + d(x, w_2). \tag{5.9}$$

Therefore, $C_d(R_D(w_1), d(w_1,x))$ and $C_{d^{-1}}(R_D(w_2), d(x,w_2))$ intersect by metric convexity of (X, d).

Furthermore, by the definition of A, for each $i, j \in I$, we see that $C_d(x_i, r_i)$ and $C_{d^{-1}}(x_j, s_j)$ intersect.

Also for each $w \in D$, $R_D(w) \in A = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$. Hence, for any $w \in D$ and $i \in I$, $C_d(R_D(w), d(w,x))$ and $C_{d^{-1}}(x_i, s_i)$ intersect, as well as for any $w \in D$ and $i \in I$, $C_{d-1}(R_D(w), d(x,w))$ and $C_d(x_i, r_i)$ intersect.

Since

$$x \in N_{\epsilon_1,\epsilon_2}(A) = \bigcap_{i\in I}(C_d(x_i, r_i + \epsilon_2) \cap C_{d^{-1}}(x_i, s_i + \epsilon_1)), \tag{5.10}$$

by Lemma 5.2 there is $a \in A$ such that $x \in C_d(a, \epsilon_2) \cap C_{d^{-1}}(a, \epsilon_1)$ and, therefore, $(C_d(x,\epsilon_1) \cap C_{d^{-1}}(x,\epsilon_2)) \cap (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$ whenever $i \in I$.

Finally, if $w \in D$, then by assumption on R_D,

$$\begin{aligned} d(R_D(w), x) &\leq d(R_D(w), w) + d(w, x) \leq \epsilon_2 + d(w, x), \\ d(x, R_D(w)) &\leq d(x, w) + d(w, R_D(w)) \leq d(x, w) + \epsilon_1. \end{aligned} \tag{5.11}$$

Thus, by metric convexity of (X, d), we have that $C_d(R_D(w), d(w,x))$ and $C_{d^{-1}}(x,\epsilon_2)$ intersect as well as $C_{d^{-1}}(R_D(w), d(x,w))$ and $C_d(x,\epsilon_1)$ intersect.

Of course, $C_d(x,\epsilon_1)$ and $C_{d^{-1}}(x,\epsilon_2)$ intersect.

We have shown that the family

$$\begin{aligned}&[C_d(R_D(w), d(w,x))_{w\in D}, (C_d(x_i, r_i))_{i\in I}, C_d(x,\epsilon_1);\\&\quad C_{d^{-1}}(R_D(w), d(x,w))_{w\in D}, (C_{d^{-1}}(x_i, s_i))_{i\in I}, C_{d^{-1}}(x,\epsilon_2)]\end{aligned}\tag{5.12}$$

of double balls has the mixed binary intersection property.

Hence, $\emptyset \neq C \subseteq A$. Now, let $u \in C$ and define $R' : D \cup \{x\} \to A$ by setting $R'(w) = R_D(w)$ if $w \in D$ and $R'(x) = u$. Then, for each $w \in D$, we have

$$\begin{aligned}d\big(R'(x), R'(w)\big) &= d(u, R_D(w)) \le d(x,w),\\ d\big(R'(w), R'(x)\big) &= d(R_D(w), u) \le d(w,x),\end{aligned}\tag{5.13}$$

so that R' is nonexpansive. Also, $d(R'(x), x) = d(u,x) \le \epsilon_2$ and $d(x, R'(x)) = d(x,u) \le \epsilon_1$. Therefore, we conclude that the pair $(D\cup\{x\}, R')$ contradicts the maximality of (D, R_D) in $(\digamma, \preccurlyeq)$. Consequently, $D = N_{\epsilon_1,\epsilon_2}(A)$ and we are done. □

Definition 5.4 (compare [19] and [20]). Let (X,d) be a T_0-quasimetric space. We say that a map $T : (X,d) \to (X,d)$ has approximate fixed points if $\inf_{x\in X} d^s(x, T(x)) = 0$.

Definition 5.5. Let (X,d) be a T_0-quasimetric space. For a map $T : (X,d) \to (X,d)$ and for any $\epsilon_1, \epsilon_2 \ge 0$, we use $F_{\epsilon_1,\epsilon_2}(T)$ to denote the set of ϵ_1,ϵ_2-approximate fixed points of T; that is, $F_{\epsilon_1,\epsilon_2}(T) = \{x \in X : d(x, T(x)) \le \epsilon_2 \text{ and } d(T(x), x) \le \epsilon_1\}$.

Theorem 5.6 (compare [13, Theorem 4.11]). *Suppose that (X,d) is a q-hyperconvex T_0-quasimetric space and that the map $T : (X,d) \to (X,d)$ is nonexpansive. Furthermore suppose that for some $\epsilon_1,\epsilon_2 \ge 0$ one has that $F_{\epsilon_1,\epsilon_2}(T)$ is nonempty. Then, the set $F_{\epsilon_1,\epsilon_2}(T)$ is q-hyperconvex.*

Proof. For each i in some nonempty index set I, let $x_i \in F_{\epsilon_1,\epsilon_2}(T)$, and let $r_i \ge 0$ and $s_i \ge 0$ satisfy

$$d\big(x_i, x_j\big) \le r_i + s_j.\tag{5.14}$$

We need to show that

$$\left[\bigcap_{i\in I}\left(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)\right)\right] \cap F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset.\tag{5.15}$$

We know that $\emptyset \neq J = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ is q-hyperconvex according to Proposition 3.1, since (X,d) is q-hyperconvex. Furthermore, J is obviously bounded in (X,d).

Also, if $x \in J$, then for each $i \in I$,

$$\begin{aligned}d(x_i, T(x)) &\le d(x_i, T(x_i)) + d(T(x_i), T(x)) \le \epsilon_2 + d(x_i, x) \le \epsilon_2 + r_i,\\ d(T(x), x_i) &\le d(T(x), T(x_i)) + d(T(x_i), x_i) \le d(x, x_i) + \epsilon_1 \le s_i + \epsilon_1.\end{aligned}\tag{5.16}$$

This proves that $T(x) \in N_{\epsilon_1,\epsilon_2}(J)$ by Lemma 5.2. Now, by Lemma 5.3, there is a nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(J)$ onto J for which $d(R(x),x) \leq \epsilon_2$ and $d(x,R(x)) \leq \epsilon_1$ whenever $x \in N_{\epsilon_1,\epsilon_2}(J)$. Also since $R \circ T$ is a nonexpansive map of J into J, it must have a fixed point by Theorem 3.3.

Suppose that $(R \circ T)(x_0) = x_0$ for some $x_0 \in J$. Then,

$$\begin{aligned} d(x_0, T(x_0)) &= d((R \circ T)(x_0), T(x_0)) \leq \epsilon_2, \\ d(T(x_0), x_0) &= d(T(x_0), (R \circ T)(x_0)) \leq \epsilon_1. \end{aligned} \tag{5.17}$$

Thus, the proof is complete, since $x_0 \in J \cap F_{\epsilon_1,\epsilon_2}(T)$. □

6. External q-Hyperconvexity

We next define an externally q-hyperconvex subset of a quasipseudometric space (X,d) in analogy to [17, Definition 3.5]. Note that this definition strengthens the concept of a q-hyperconvex subset of (X,d) (compare also [21, Definition 3]).

Definition 6.1. Let (X,d) be a quasipseudometric space. A subspace E of (X,d) is said to be externally q-hyperconvex (relative to X) if given any family $(x_i)_{i\in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds:

if $d(x_i,x_j) \leq r_i + s_j$ whenever $i,j \in I$, $\operatorname{dist}(x_i,E) \leq r_i$ and $\operatorname{dist}(E,x_i) \leq s_i$ whenever $i \in I$, then $\bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i)) \cap E \neq \emptyset$.

In the following, $\mathcal{E}_q(X)$ will denote the set of nonempty externally q-hyperconvex subsets of (X,d).

Example 6.2 (compare [21, Theorem 7]). Let E be a nonempty externally q-hyperconvex subset in a quasipseudometric space (X,d) and let x be any point of X. Set $\operatorname{dist}(x,E) = r$ and $\operatorname{dist}(E,x) = s$. Then, by applying external q-hyperconvexity of E to the double ball $(C_d(x,r); C_{d^{-1}}(x,s))$, we conclude that there is $p \in C_d(x,r) \cap C_{d^{-1}}(x,s) \cap E$. Thus, $d(x,p) = \operatorname{dist}(x,E)$ and $d(p,x) = \operatorname{dist}(E,x)$.

Lemma 6.3 (compare [17, Lemma 3.8]). *Let (X,d) be a q-hyperconvex space and let $x \in X$. Furthermore, let $\emptyset \neq A = \bigcap_{i\in I}(C_d(x_i,r_i) \cap C_{d^{-1}}(x_i,s_i))$ where $(x_i)_{i\in I}$ is a nonempty family of points in X and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ are families of nonnegative reals. Then, there is $p \in A$ such that* $\operatorname{dist}(x,A) = d(x,p)$ *and* $\operatorname{dist}(A,x) = d(p,x)$.

Proof. Evidently,

$$\begin{aligned} &[(C_d(x_i,r_i))_{i\in I}, (C_d(x, \operatorname{dist}(x,A) + \epsilon))_{\epsilon>0}; \\ &\quad (C_{d^{-1}}(x_i,s_i))_{i\in I}, (C_{d^{-1}}(x, \operatorname{dist}(A,x) + \epsilon))_{\epsilon>0}] \end{aligned} \tag{6.1}$$

satisfies the mixed binary intersection property. Thus, there is

$$p \in A \cap C_d(x, \operatorname{dist}(x, A)) \cap C_{d^{-1}}(x, \operatorname{dist}(A, x)) \tag{6.2}$$

by q-hyperconvexity of (X, d). Obviously, p then satisfies the stated condition. □

The following lemma, which makes use of Lemma 6.3, will be useful in the proof of Theorem 6.5. Considering the case that $E = X$, we see that Lemma 6.4 improves on Proposition 3.1.

Lemma 6.4 (compare [18, Lemma 2]). *Let (X, d) be a q-hyperconvex quasipseudometric space. Suppose that $E \subseteq X$ is externally q-hyperconvex relative to X and suppose that A is a q-admissible subset of (X, d) such that $E \cap A \neq \emptyset$. Then $E \cap A$ is externally q-hyperconvex relative to X.*

Proof. Assume that a given nonempty family $(x_\alpha)_{\alpha\in S}$ of points in X and families of nonnegative real numbers $(r_\alpha)_{\alpha\in S}$ and $(s_\alpha)_{\alpha\in S}$ satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$, $\operatorname{dist}(x_\alpha, A \cap E) \leq r_\alpha$, and $\operatorname{dist}(A \cap E, x_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in S$.

Since A is q-admissible, $\emptyset \neq A = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ with $x_i \in X$ and $r_i, s_i \geq 0$ whenever $i \in I$. Because $\operatorname{dist}(x_\alpha, A) \leq \operatorname{dist}(x_\alpha, A \cap E) \leq r_\alpha$ and $\operatorname{dist}(A, x_\alpha) \leq \operatorname{dist}(A \cap E, x_\alpha) \leq s_\alpha$ whenever $\alpha \in S$, it follows that for each $\alpha \in S$, $i \in I$ and for $p \in A$ chosen according to Lemma 6.3 we have

$$\begin{aligned} d(x_\alpha, x_i) &\leq d(x_\alpha, p) + d(p, x_i) \leq r_\alpha + s_i, \\ d(x_i, x_\alpha) &\leq d(x_i, p) + d(p, x_\alpha) \leq r_i + s_\alpha. \end{aligned} \tag{6.3}$$

Also, since for each $i \in I$, $A \subseteq C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)$, and since $A \cap E \neq \emptyset$, it follows that

$$\begin{aligned} &\operatorname{dist}(x_i, E) \leq r_i, \\ &\operatorname{dist}(E, x_i) \leq s_i, \end{aligned} \tag{6.4}$$

and that $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. Trivially, we have $\operatorname{dist}(x_\alpha, E) \leq r_\alpha$ and $\operatorname{dist}(E, x_\alpha) \leq s_\alpha$ whenever $\alpha \in S$.

Therefore, by external q-hyperconvexity of E, we conclude that

$$\begin{aligned} &\left[\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))\right] \cap \left[\bigcap_{\alpha\in S}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap E\right] \\ &\quad = \bigcap_{\alpha\in S}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap (E \cap A) \neq \emptyset. \end{aligned} \tag{6.5}$$

Thus, the proof is complete. □

We next show that the intersection of a descending family of externally q-hyperconvex nonempty subspaces of a bounded q-hyperconvex T_0-quasimetric space behaves as expected.

Theorem 6.5 (compare [18, Theorem 4]). *Let (X, d) be a bounded q-hyperconvex T_0-quasimetric space. Moreover, let $(X_i)_{i\in I}$ be a descending family of nonempty externally q-hyperconvex subsets of X, where I is assumed to be totally ordered such that $i_1, i_2 \in I$ and $i_1 \leq i_2$ if and only if $X_{i_2} \subseteq X_{i_1}$. Then $\bigcap_{i\in I} X_i$ is nonempty and externally q-hyperconvex relative to X.*

Proof. Theorem 4.1 implies that $D = \bigcap_{i\in I} X_i \neq \emptyset$. In order to show that D is externally q-hyperconvex, let a nonempty family $(x_\alpha)_{\alpha\in S}$ of points in X and families of nonnegative real numbers $(r_\alpha)_{\alpha\in S}$ and $(s_\alpha)_{\alpha\in S}$ be given such that $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$, and $\text{dist}(x_\alpha, D) \leq r_\alpha$ and $\text{dist}(D, x_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in S$.

Since X is q-hyperconvex, we know that $A := \bigcap_{\alpha\in S}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \neq \emptyset$. Also, since for each $\alpha \in S$, $\text{dist}(x_\alpha, D) \leq r_\alpha$ and $\text{dist}(D, x_\alpha) \leq s_\alpha$, we have $\text{dist}(x_\alpha, X_i) \leq r_\alpha$ and $\text{dist}(X_i, x_\alpha) \leq s_\alpha$ whenever $i \in I$, so that, by external q-hyperconvexity of X_i, we conclude that $A \cap X_i \neq \emptyset$ whenever $i \in I$.

By Lemma 6.4, $(A \cap X_i)_{i\in I}$ is a descending chain of nonempty (externally) q-hyperconvex subsets of (X, d), so that again by Theorem 4.1 $\bigcap_{i\in I}(A \cap X_i) = A \cap D \neq \emptyset$. □

Let us note that the result stated in our abstract is a consequence of our next theorem.

Theorem 6.6 (compare [18, Theorem 1]). *Let (H, d) be a q-hyperconvex T_0-quasimetric space, let X be any set, and let a map $T^* : X \to \mathcal{E}_q(H)$ be given. Then, there exists a map $T : X \to H$ for which $T(x) \in T^*(x)$ whenever $x \in X$ and for which $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ whenever $x, y \in X$.*

Proof. Let $\mathcal{F}$ denote the collection of all pairs (D, T), where $D \subseteq X, T : D \to H, T(d) \in T^*(d)$ whenever $d \in D$, and $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ whenever $x, y \in D$. Note that $\mathcal{F} \neq \emptyset$, since $(\{x_0\}, T) \in \mathcal{F}$ for any choice of $x_0 \in X$ and $T(x_0) \in T^*(x_0)$. Define a partial order relation on $\mathcal{F}$ by setting $(D_1, T_1) \preccurlyeq (D_2, T_2)$ if and only if $D_1 \subseteq D_2$, and, $T_2|_{D_1} = T_1$.

Let $((D_\alpha, T_\alpha))_{\alpha\in S}$ be an increasing chain in $(\mathcal{F}, \preccurlyeq)$. Then it follows that $(\bigcup_{\alpha\in S} D_\alpha, T) \in \mathcal{F}$ where $T|D_\alpha = T_\alpha$. By Zorn's lemma, $(\mathcal{F}, \preccurlyeq)$ has a maximal element, say (D, T). Assume that $D \neq X$ and select $x_0 \in X \setminus D$. Set $\widetilde{D} = D \cup \{x_0\}$ and consider the set

$$J = \bigcap_{x\in D} [C_d(T(x), d_H(T^*(x), T^*(x_0))) \cap C_{d^{-1}}(T(x), d_H(T^*(x_0), T^*(x)))] \cap T^*(x_0). \tag{6.6}$$

Since $T^*(x_0) \in \mathcal{E}_q(H)$, by definition of external q-hyperconvexity, $J \neq \emptyset$ if for each $x \in D$, we have $\text{dist}(T(x), T^*(x_0)) \leq d_H(T^*(x), T^*(x_0))$ and

$$\text{dist}(T^*(x_0), T(x)) \leq d_H(T^*(x_0), T^*(x)), \tag{6.7}$$

and for any $x, y \in D$, also

$$d(T(x), T(y)) \leq d_H(T^*(x), T^*(x_0)) + d_H(T^*(x_0), T^*(y)). \tag{6.8}$$

We are going to check that these conditions hold.

Let $x \in D$. For each $\epsilon > 0$, we have $T^*(x) \subseteq B_{d^{-1}}(T^*(x_0), d_H(T^*(x), T^*(x_0)) + \epsilon)$ and $T^*(x) \subseteq B_d(T^*(x_0), d_H(T^*(x_0), T^*(x)) + \epsilon)$ by definition of the Hausdorff quasipseudometric.

Since $T(x) \in T^*(x)$, for each $\epsilon > 0$, there is $a \in T^*(x_0)$ such that $d(T(x), a) \leq d_H(T^*(x), T^*(x_0)) + \epsilon$, and there is $b \in T^*(x_0)$ such that

$$d(b, T(x)) \leq d_H(T^*(x_0), T^*(x)) + \epsilon. \tag{6.9}$$

Therefore, $\operatorname{dist}(T(x), T^*(x_0)) \leq d_H(T^*(x), T^*(x_0))$ and $\operatorname{dist}(T^*(x_0), T(x)) \leq d_H(T^*(x_0), T^*(x))$.

We finally also note that by assumption on T, for each $x, y \in D$ we have that

$$d(T(x), T(y)) \leq d_H(T^*(x), T^*(y)) \leq d_H(T^*(x), T^*(x_0)) + d_H(T^*(x_0), T^*(y)). \tag{6.10}$$

Thus, we have shown that $J \neq \emptyset$. Choose $y_0 \in J$ and define

$\widetilde{T}(x) = y_0$ if $x = x_0$ and $\widetilde{T}(x) = T(x)$ if $x \in D$.

Since for each $x \in D$, $d(\widetilde{T}(x_0), \widetilde{T}(x)) = d(y_0, T(x)) \leq d_H(T^*(x_0), T^*(x))$ and $d(\widetilde{T}(x), \widetilde{T}(x_0)) = d(T(x), y_0) \leq d_H(T^*(x), T^*(x_0))$, we conclude that $(D \cup \{x_0\}, \widetilde{T}) \in \mathcal{F}$, contradicting the maximality of (D, T). Therefore, $D = X$. □

Corollary 6.7 (compare [18, Corollary 1]). *Let (H, d) be a q-hyperconvex T_0-quasimetric space. Moreover, let (X, ρ) be a T_0-quasimetric space, and suppose that $T^* : X \to \mathcal{E}_q(H)$ is nonexpansive, that is, $d_H(T^*(x), T^*(y)) \leq \rho(x, y)$ whenever $x, y \in X$. Then, there is a nonexpansive map $T : (X, \rho) \to (H, d)$ for which $T(x) \in T^*(x)$ whenever $x \in X$.*

Proof. Because T^* is nonexpansive, the selection obtained from Theorem 6.6 is also nonexpansive. □

Corollary 6.8 (compare [18, Corollary 2]). *Let H be a bounded and q-hyperconvex T_0-quasimetric space and suppose that $T^* : H \to \mathcal{E}_q(H)$ is nonexpansive. Then T^* has a fixed point, that is, there exists $x \in H$ such that $x \in T^*(x)$.*

Proof. The existence of a fixed point for the nonexpansive selection T of T^*, which exists by Corollary 6.7, follows from Theorem 3.3. □

In the following theorem, we set $\operatorname{Fix}(T^*) = \{x \in H : x \in T^*(x)\}$. According to Corollary 6.8, $\operatorname{Fix}(T^*) \neq \emptyset$ if H is bounded and q-hyperconvex, and T^* is nonexpansive.

Theorem 6.9 (compare [18, Theorem 2]). *Let (H, d) be a q-hyperconvex T_0-quasimetric space, let $T^* : H \to \mathcal{E}_q(H)$ be a nonexpansive map and suppose that* $\operatorname{Fix}(T^*) \neq \emptyset$. *Then, there exists a nonexpansive map $T : H \to H$ with $T(x) \in T^*(x)$ whenever $x \in H$ and for which* $\operatorname{Fix}(T) = \operatorname{Fix}(T^*)$.

Proof. Let $\mathcal{F}$ denote the collection of all pairs (D, T), where $\operatorname{Fix}(T^*) \subseteq D \subseteq H, T : D \to H, T(d) \in T^*(d)$ whenever $d \in D$, $T(x) = x$ whenever $x \in \operatorname{Fix}(T^*)$, and $d(T(x), T(y)) \leq d(x, y)$ whenever $x, y \in D$. By assumption $(\operatorname{Fix}(T^*), \mathrm{Id}) \in \mathcal{F}$, so $\mathcal{F} \neq \emptyset$. The argument now follows from a modification of the proof of Theorem 6.6. One defines a partial order on $\mathcal{F}$ by setting $(D_1, T_1) \preccurlyeq (D_2, T_2)$ if and only if $D_1 \subseteq D_2$ and $T_2|_{D_1} = T_1$.

Let $((D_\alpha, T_\alpha))_{\alpha\in S}$ be an increasing chain in $(\mathcal{F}, \preccurlyeq)$. Then, it follows that $(\bigcup_{\alpha\in S} D_\alpha, T) \in \mathcal{F}$ where $T|_{D_\alpha} = T_\alpha$ whenever $\alpha \in S$. By Zorn's lemma, $(\mathcal{F}, \preccurlyeq)$ has a maximal element, say (D, T). Assume $D \neq H$ and find $x_0 \in H \setminus D$. Set $\widetilde{D} = D \cup \{x_0\}$ and consider the set:

$$J = \bigcap_{x\in D} [C_d(T(x), d(x, x_0)) \cap C_{d^{-1}}(T(x), d(x_0, x))] \cap T^*(x_0). \tag{6.11}$$

Since $T^*(x_0) \in \mathcal{E}_q(H)$, by definition of external q-hyperconvexity, $J \neq \emptyset$ if for each $x \in D$, we have $\operatorname{dist}(T(x), T^*(x_0)) \leq d(x, x_0)$ and $\operatorname{dist}(T^*(x_0), T(x)) \leq d(x_0, x)$, and for any $x, y \in D$ we have $d(T(x), T(y)) \leq d(x, x_0) + d(x_0, y)$.

We are going to check these conditions next. Let $x \in D$. For each $\epsilon > 0$, we have $T^*(x) \subseteq B_{d^{-1}}(T^*(x_0), d_H(T^*(x), T^*(x_0)) + \epsilon) \subseteq B_{d^{-1}}(T^*(x_0), d(x, x_0) + \epsilon)$ and $T^*(x) \subseteq B_d(T^*(x_0), d_H(T^*(x_0), T^*(x)) + \epsilon) \subseteq B_d(T^*(x_0), d(x_0, x) + \epsilon)$ by definition of the Hausdorff quasipseudometric.

Since $T(x) \in T^*(x)$, for each $\epsilon > 0$, there is $a \in T^*(x_0)$ such that $d(T(x), a) \leq d(x, x_0) + \epsilon$, and there is $b \in T^*(x_0)$ such that $d(b, T(x)) \leq d(x_0, x) + \epsilon$. Therefore, $\operatorname{dist}(T(x), T^*(x_0)) \leq d(x, x_0)$ and $\operatorname{dist}(T^*(x_0), T(x)) \leq d(x_0, x)$.

We finally also note that by assumption on T for each $x, y \in D$ we have that $d(T(x), T(y)) \leq d(x, y) \leq d(x, x_0) + d(x_0, y)$.

Thus, we have shown that $J \neq \emptyset$. Choose $y_0 \in J$ and define $\widetilde{T}(x) = y_0$ if $x = x_0$ and $\widetilde{T}(x) = T(x)$ if $x \in D$.

Since for each $x \in D, d(\widetilde{T}(x_0), \widetilde{T}(x)) = d(y_0, T(x)) \leq d(x_0, x)$ and

$$d\left(\widetilde{T}(x), \widetilde{T}(x_0)\right) = d(T(x), y_0) \leq d(x, x_0), \tag{6.12}$$

we conclude that $(D \cup \{x_0\}, \widetilde{T}) \in \mathcal{F}$, contradicting the maximality of (D, T). Therefore, $D = H$. □

We will next establish the q-hyperconvexity of the space of all bounded λ-Lipschitzian self-maps on a q-hyperconvex T_0-quasimetric space.

Theorem 6.10 (compare [18, Theorem 3]). *Let (X, d) be a q-hyperconvex T_0-quasimetric space and for $\lambda > 0$ let $\mathcal{F}_\lambda$ denote the family of all bounded λ-Lipschitzian self-maps on (X, d) equipped with the T_0-quasimetric $\widehat{d}(f, g) = \sup_{x\in X} d(f(x), g(x))$ whenever $f, g \in \mathcal{F}_\lambda$. Then $(\mathcal{F}_\lambda, \widehat{d})$ is itself a q-hyperconvex T_0-quasimetric space.*

Proof. We leave it to the reader to verify that $\widehat{d}$ is an extended T_0-quasimetric on the set $\mathcal{F}_\lambda$. We next note that $\widehat{d}$ is a T_0-quasimetric, since $\widehat{d}$ does not attain ∞. Indeed, let $x_0,\ x \in X$ and $f, g \in \mathcal{F}_\lambda$. Then,

$$\begin{aligned} d(f(x), g(x)) &\leq |d(f(x), g(x)) - d(f(x_0), g(x_0))| + d(f(x_0), g(x_0)) \\ &\leq d^s(f(x_0), f(x)) + d^s(g(x), g(x_0)) + d(f(x_0), g(x_0)) \\ &\leq M_f + M_g + d(f(x_0), g(x_0)) \end{aligned} \tag{6.13}$$

for some positive real constants M_f and M_g, since f and g are bounded. We conclude that $\widehat{d}(f,g) \neq \infty$.

Suppose that $(f_\alpha)_{\alpha\in S}$ is a nonempty family of functions in $\mathcal{F}_\lambda$ and let

$$(r_\alpha)_{\alpha\in S}, (s_\alpha)_{\alpha\in S}, \tag{6.14}$$

be families of nonnegative reals such that $\widehat{d}(f_\alpha, f_\beta) \leq r_\alpha + s_\beta$ whenever $\alpha, \beta \in S$. Then, for each $x \in X$, we have $d(f_\alpha(x), f_\beta(x)) \leq r_\alpha + s_\beta$ whenever $\alpha, \beta \in S$. So because of the q-hyperconvexity of (X, d), we have that

$$J(x) = \bigcap_{\alpha\in S} \left(C_d(f_\alpha(x), r_\alpha) \cap C_{d^{-1}}(f_\alpha(x), s_\alpha) \right) \neq \emptyset. \tag{6.15}$$

Note that, by Lemma 6.4 applied to $E = X$, we see that $J(x) \in \mathcal{E}_q(X)$ whenever $x \in X$.

We next show that $d_H(J(x), J(y)) \leq \lambda d(x,y)$ whenever $x, y \in X$. To this end, it suffices to show that for each $x, y \in X$ we have $J(y) \subseteq C_d(J(x), \lambda d(x,y))$ (i.e., $J(x) \subseteq C_d(J(y), \lambda d(y,x))$, and that $J(x) \subseteq C_{d^{-1}}(J(y), \lambda d(x,y))$.

Fix $x, y \in X$. If $z \in J(x)$, then for each $\alpha \in S$, by the λ-Lipschitzian condition satisfied by f_α,

$$\begin{gathered} d(z, f_\alpha(y)) \leq d(z, f_\alpha(x)) + d(f_\alpha(x), f_\alpha(y)) \leq d(z, f_\alpha(x)) + \lambda d(x,y) \leq s_\alpha + \lambda d(x,y), \\ d(f_\alpha(y), z) \leq d(f_\alpha(y), f_\alpha(x)) + d(f_\alpha(x), z) \leq \lambda d(y,x) + r_\alpha. \end{gathered} \tag{6.16}$$

By Lemma 5.2 applied to $N_{\lambda d(x,y), \lambda d(y,x)}(J(y))$, we then have

$$\begin{aligned} z &\in \bigcap_{\alpha\in S} \left[C_d(f_\alpha(y), r_\alpha + \lambda d(y,x)) \cap C_{d^{-1}}(f_\alpha(y), s_\alpha + \lambda d(x,y)) \right] \\ &= N_{\lambda d(x,y), \lambda d(y,x)}(J(y)) = \bigcup_{a\in J(y)} \left[C_d(a, \lambda d(y,x)) \cap C_{d^{-1}}(a, \lambda d(x,y)) \right]. \end{aligned} \tag{6.17}$$

Therefore, $J(x) \subseteq C_{d^{-1}}(J(y), \lambda d(x,y))$, and $J(x) \subseteq C_d(J(y), \lambda d(y,x))$ and thus $J(y) \subseteq C_d(J(x), \lambda d(x,y))$ whenever $x, y \in X$. Hence, our claim is verified.

In the light of Theorem 6.6 for each $x \in X$, it is possible to find $f(x) \in J(x)$ so that we get $f \in \mathcal{F}_\lambda$, since $d(f(x), f(y)) \leq d_H(J(x), J(y)) \leq \lambda d(x,y)$ whenever $x, y \in X$. In particular, we also note that f is bounded. Indeed, fix $\alpha \in S$. Then, for any $x, y \in X$ and some positive real constant M_{f_α}, we have $d(f(x), f(y)) \leq d(f(x), f_\alpha(x)) + d(f_\alpha(x), f_\alpha(y)) + d(f_\alpha(y), f(y)) \leq s_\alpha + M_{f_\alpha} + r_\alpha$ by the choice of f. Thus, f is indeed bounded.

Since $f \in \bigcap_{\alpha\in S}(C_{\widehat{d}}(f_\alpha, r_\alpha) \cap C_{\widehat{d}^{-1}}(f_\alpha, s_\alpha))$, we have shown that $(\mathcal{F}_\lambda, \widehat{d})$ is q-hyperconvex. □

We conclude this article with a curious observation in the spirit of [18, Proposition 2].

Proposition 6.11. *Suppose that (X, d) is a bounded q-hyperconvex T_0-quasimetric space and let $U = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ and $V = \bigcap_{i\in I}(C_d(y_i, r_i) \cap C_{d^{-1}}(y_i, s_i))$ with two nonempty families $(x_i)_{i\in I}, (y_i)_{i\in I}$ of points in X and two families $(r_i)_{i\in I}, (s_i)_{i\in I}$ of nonnegative reals. Then, $d_H(V, U) \leq \sup\{d(y_i, x_i) : i \in I\}$.*

Proof. Let $\rho_{UV} = \sup\{d(x_i, y_i) : i \in I\}$ and similarly, let $\rho_{VU} = \sup\{d(y_i, x_i) : i \in I\}$, and let $x \in U$. Then, for each $i \in I$, $d(x, y_i) \leq d(x, x_i) + d(x_i, y_i) \leq s_i + \rho_{UV}$ and $d(y_i, x) \leq d(y_i, x_i) + d(x_i, x) \leq \rho_{VU} + r_i$. Consequently, $x \in \bigcap_{i\in I}(C_d(y_i, r_i + \rho_{VU}) \cap C_{d^{-1}}(y_i, s_i + \rho_{UV})) = \bigcup_{a\in V}(C_d(a, \rho_{VU}) \cap C_{d^{-1}}(a, \rho_{UV}))$ by Lemma 5.2.

Therefore, $U \subseteq C_d(V, \rho_{VU})$ and $U \subseteq C_{d^{-1}}(V, \rho_{UV})$, and similarly, by interchanging U and V, hence, $V \subseteq C_{d^{-1}}(U, \rho_{VU})$. We have shown that $d_H(V, U) \leq \rho_{VU}$. □

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