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Research Article

Pricing Multivariate European Equity Option Using Gaussians Mixture Distributions and EVT-Based Copulas

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In this article, we present an approach which allows taking into account the effect of extreme values in the modeling of financial asset returns and in the valorisation of associated options. Specifically, the marginal distribution of asset returns is modelled by a mixture of two Gaussian distributions. Moreover, we model the joint dependence structure of the returns using a copula function, the extremal one, which is suitable for our financial data, particularly the extreme values copulas. Applications are made on the Atos and Dassault Systems actions of the CAC40 index. Monte Carlo method is used to compute the values of some equity options such as the call on maximum, the call on minimum, the digital option, and the spreads option with the basket (Atos, Dassault systems) as underlying.

1. Introduction

Since the pioneering work of Black and Scholes [1] and Cox et al. [2] (respectively, in the continuous and discrete case), option pricing has become a crucial topic in finance. Indeed, considering a European-type option on an underlying asset with a price S_t, strike K, and expiration T, Black and Scholes have made it possible to determine a formula for the price of such options under certain assumptions, the fundamentals of which are the lack of arbitrage opportunity and that on the price S_t of the asset underlying (S_t follows geometric Brownian motion), i.e.,

$$dS_t = \mu S_t dt + \sigma S_t W_t, \tag{1}$$

where μ and σ are constant and W_t is a standard geometric Brownian motion.

Thus, the formulas of the theoretical relative values of a call option and put option are derived.

Options are essential financial products allowing to their holders to hedge against the risk of their investments falling. This is how we are increasingly seeing the creation of several types of options such as exotic options, multivariate options, etc., with the aim of providing more security. As a result, valuation models are also evolving. Of all the multiple option pricing models, it turns out that each one is primarily based on the dynamic of underlying asset pricing model (for options with only one underlying) or the asset portfolio (for options on multiple assets), when market assumptions are known. In fact, since the assumption of no arbitrage opportunity (NAO) in the markets is the basis of the fundamental results obtained in finance, it is considered by default (there are markets on which the arbitration assumption is considered). The advantage under this NAO assumption is that, associated with that of market completeness, there is a single risk-neutral probability for which the discounted flows are martingales. In the univariate case, one of the most interesting results obtained in this direction on valuation is that of Breeden et al. [3]. It states that the second derivative (when it exists and is continuous) of the price of a standard option relative to the strike coincides with the risk-neutral density. Indeed, if D_t is the price of a European option of an underlying asset with price X_t having for pay-off $g(X_T)$, T

the time to expiration, and r the risk-free interest rate, then the risk-neutral density $f^*(X_T)$ is linked to D_t by

$$D_t = e^{-r(T-t)}\mathbb{E}^*\{g(X_T)\} = e^{-r(T-t)}\int g(X_T)f^*(X_T)\mathrm{d}X_T. \tag{2}$$

For valuation in the multivariate framework, this risk-neutral formula is a simple generalization (for example, [4, 5] are used this generalization). Talponen and Viitasaari [6] recently gave the multivariate version of the univariate result.

Multivariate options (rainbow, digital, quantos, etc.), which will be the main subject of our study in this paper, constitute the central themes of current research on financial risk coverage (see Tsuzuki [7]). The advantage lies in the fact that they offer better coverage against risks. Indeed, the basic idea is that when the option is a function of several assets, the fall in value of one asset is compensated by the rise of another asset in the portfolio. Thus, the association or dependence between assets plays a major role in the pricing of these types of options. To take such an aspect into account in the valuation, the use of the copula is a good alternative.

The valuation of multivariate options by copulas is in full development. The copula gives the advantage of joining the marginal and the dependence structure. This is the case for many works on the valuation of options with copulas; the emphasis is first on marginal risk-neutral densities and then on the joint risk-neutral density (risk-neutral copula). For example, we can cite the work of Cherubuni and Luciano [8], Cherubuni and Luciano [9], Rosenberg [10], Salmon and Schleichere [11], and Slavchev and Wilkens [12]. However, all this work did not take into account the effects of extreme values in the marginal, which is not without effect on valuation (risk of overvaluation or undervaluation). However, there are other copula modeling approaches based on volatility dynamics as in van den Goorbergh et al. [13], Bernard and Czado [14], and Barban and Di Persio [4]. The reader can consult them for full details on the literature on this approach.

In this present study, we propose a valuation method for multivariate options allowing taking into account the effects of extreme values in the marginal and the joint structure on the basis of the works of Idier et al. [15] and the use of extreme values copulas.

In the rest of this work, in the first section we give the results obtained by Idier et al. [15] which will be necessary and some essential notions on copulas. In the second section, we expose the methodology used for leading properly the application of the approach. Then, the obtained results of different estimations and simulations are presented, with their analysis and interpretations. The last section presents a conclusion and discussion.

2. Preliminaries

2.1. Results of an Approach of Modeling Financial Assets. It is proved that the empirical distribution of financial asset returns has thicker tails than that of the Gaussian distribution. This indicates the presence of extreme values. This fact shows also that the normal distribution does not make it possible to model rigorously the returns of financial assets because it does not take into account the extreme. This is the case with the method proposed by Black and Scholes [1].

To take into account the effects of extreme values, Idier et al. [15] proposed, as an alternative to the normal distribution, modeling the distribution of the rates of return of the underlying asset of an univariate option, under NAO assumption, by a mixture of Gaussian distribution in the continuous framework (their method is a generalization of the method in the discrete case of Bertholon, Monfort and Pegoraro, Pegoraro). They justified their choice by the fact that a mixture of Gaussian distributions makes it possible to approximate all the distributions usually used (Gaussian, alpha-stable, Student, hyperbolic, etc.); also, it has certain theoretical properties allowing easy handling in the frame of theoretical model for valuing asset price and it is easy to simulate and can reproduce various sets (mean, variance, skewness, and kurtosis) observed in the data.

Under the assumption that the historical distribution of the returns of the underlying $X_{t+1} = \ln(S_{t+1}/S_t)$ where S_t is the price at time t of the underlying asset is a mixture of 2 Gaussian distributions, its density is given by

$$f(x) = \sum_{i=1}^{2} p_i.n\left(x, \mu_i, \sigma_i^2\right), \tag{3}$$

where

$$n\left(x, \mu_i, \sigma_i^2\right) = \frac{1}{\sigma_i\sqrt{2\pi}}\exp\left\{\frac{-(x-u_i)^2}{2\sigma_i^2}\right\} \tag{4}$$

is the density of a Gaussian distribution with mean μ_i and standard deviation σ_i; $0 < p_i < 1$ and $\sum_{i=1}^{2} p_i = 1$.

Moreover, the stochastic discount factor is characterized by an affine exponential form, i.e.,

$$M_{t;t+1} = \exp\{\alpha_t X_t + \beta_t\}. \tag{5}$$

They establish, under these assumptions, that the risk-neutral distribution is also a Gaussian mixture and that its density f^* is defined by

$$f^*(x) = \sum_{i=1}^{2} v_i.n\left(x, \mu_i + \alpha.\sigma_i^2, \sigma_i^2\right), \tag{6}$$

where

$$v_i = \frac{\left(p_i \exp\left\{\mu_i\alpha + \left(\alpha^2/2\right)\sigma_i^2\right\}\right)}{\left(\sum_{i=1}^{2} p_i \exp\ \mu_i\alpha + \left(\alpha^2/2\right)\sigma_i^2\right)}, \tag{7}$$

with $0 < v_i < 1, \sum_{i=1}^{2} v_i = 1$.

Thus, they derive the relative theoretical price of a European call with a one-period maturity $(T = t + 1)$ and a relative strike k:

$$c_t(T,k) = \sum_{i=1}^{2} v_i\gamma_i c_{\mathrm{bs}}\left(\sigma_i^2, \frac{k}{\gamma_i}\right), \tag{8}$$

where $c_{\mathrm{bs}}(.,.)$ is the Black–Scholes one-period $(T = t + 1)$ formula for the relative price of a call and $\gamma_i = \exp\{\mu_i + \alpha.\sigma_i^2 - r + (\sigma_i^2/2)\}$, for $J = 2$.

Remark 1. The existence of the call-put parity relation makes it possible to simplify the task in calculating option price. It is then sufficient to calculate the price of the call to deduce that of the corresponding put (or vice versa) by the relation

$$C_t(T,K) + Ke^{-r(T-t)} = P_t(T,K) + S_t. \tag{9}$$

2.2. A Survey of Copulas. In this section, we recall the basics on copulas. These are the definitions and properties essential for our study. For more details on copulas, see Nelsen [16].

2.3. Definitions and Properties. The copula is a function allowing capturing the structure of dependence between several random variables.

A function $C\colon [0,1]^d \longrightarrow [0,1]$ is a d-copula if it satisfies the following properties:

(i) For all u in $[0,1]$, $C(1,\ldots,1,u,1,\ldots,1) = u$.

(ii) For all u_i in $[0,1]$, $C(u_1,\ldots,u_d) = 0$ if at least one of the u_i is zero.

(iii) C is "grounded" and d-increasing, i.e.,

$$\sum_{i_1=1}^{2}\cdots\sum_{i_d=1}^{2}(-1)^{i_1+\cdots+i_d}C\left(u_{1,i_1},\ldots,u_{d,i_d}\right) \geq 0, \tag{10}$$

for all $(u_{1,1},\ldots,u_{d,1})$ and $(u_{1,2},\ldots,u_{d,2}) \in [0,1]^d$ with $u_{d,1} \leq u_{d,2}$.

The fundamental result on the copula due to Sklar states that for a whole multivariate distribution F with continuous marginal $F_1,\ldots,F_d$, there exists a unique (uniqueness is not guaranteed when marginal is not continuous) copula $C\colon [0,1]^d \longrightarrow [0,1]$ such that

$$F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d)). \tag{11}$$

Conversely, when C is a copula and $F_1,\ldots,F_d$ are marginal distributions, the function F defined by (11) is a multivariate distribution with marginal distributions $F_1,\ldots,F_d$.

This result makes it possible to deduce several properties of the copula including invariance by any monotonic transformation. Another consequence of Sklar's theorem is that every copula C satisfies

$$\max\left(\sum_{i=1}^{d}u_i - d + 1; 0\right) \leq C(u_1,u_2,\ldots,u_d) \leq \min(u_1,u_2,\ldots,u_d). \tag{12}$$

This relation is the variant in terms of copulas of the Frechet–Hoeffding bounds of a multivariate distribution. The upper bound $\min(u_1,u_2,\ldots,u_d)$ is the comonotonic copula representing the perfect positive dependence. The lower bound $\max(\sum_{i=1}^{d}u_i - d + 1; 0)$ is a copula only for $d = 2$. In this case, it represents the perfect negative dependence.

Remark 2. If $\overline{F}$ is the multivariate survival distribution of a F distribution of marginal $F_i; i = 1,\ldots,d$, then the survival copula, denoted by $\overline{C}$, is defined by

$$\overline{F}(x_1,\ldots,x_d) = \overline{C}(\overline{F}_1(x_1),\ldots,\overline{F}_d(x_d)). \tag{13}$$

The survival copula $\overline{C}$ is related to the copula C, for all $(u_1,u_2,\ldots,u_d) \in [0,1]^d$, by

$$\overline{C}(u_1,u_2,\ldots,u_d) = \sum_{M\subset N}(-1)^m C\left((1-u_1)^{1_1\in M},(1-u_2)^{1_2\in M},\ldots,(1-u_d)^{1_d\in M}\right), \tag{14}$$

where $N = \{1,2,\ldots,d\}$, $m = |M|$ is the cardinality of M, and $1_i \in M$ indicates that i belongs to M.

It is therefore advisable not to confuse the dual copula with the survival copula.

2.4. Sample of Copulas for Finance in This Study

2.4.1. Archimedean Copulas. In the literature, there are several families of copulas, some of which are more suited to financial modeling. Archimedean copulas family includes the models of Clayton, Frank, and Gumbel. These copulas have the advantage of capturing the structure of positive or negative dependence between the variables. These types of dependences are characteristics of financial variables, which justifies the use of this copula family. In terms of option pricing, for example, these copulas have been used in Cherubuni and Luciano [8] and in Slavchev and Wilkens [12].

2.4.2. Elliptical Copulas. Other types of copulas used in finance are the normal copula and the t-copula. They belong to the family of elliptical copulas which describe the dependence structure of elliptical distributions. The choice of this family is justified by the fact that elliptic distributions have long been used to model random phenomena in many fields. Despite the demonstration of the leptokurtic character of the returns of financial series, of which they have the weakness to rigorously model, they are still used.

2.5. Estimation-Adequacy Test of a Copula. The choice of the copula rigorously describing multivariate statistical data requires estimation and conformity testing. There are several techniques in the literature for estimating copulas belonging to different families: parametric, semiparametric, and nonparametric. For more details on these methods, see Bouyé [17].

2.5.1. Estimation by the IFM Method. The IFM (inference functions of margins) method is a two-step estimation method of a copula. It was presented by Shih and Louis [18] in the bivariate case and then developed in dimension greater than two by Joe and Xu [19]. It is carried out as follows:

(1) The first step consists in finding the estimators $\widehat{\alpha}_i$ of the parameters $\alpha_i, i = 1, \ldots, d$ for marginal distributions by maximum likelihood:

$$\widehat{\alpha}_i = \arg\max \sum_{j=1}^{N} \ln f_n\left(x_i^j; \alpha_i\right). \tag{15}$$

(2) Once the marginals have been determined, we estimate the parameter θ of the copula that best describes these marginals by the maximum likelihood.

$$\widehat{\theta} = \arg\max \sum_{j=1}^{N} \ln c\left(F_1\left(x_1^j; \widehat{\alpha}_1\right); \ldots; F_d\left(x_d^j; \widehat{\alpha}_d\right); \theta\right). \tag{16}$$

One of the advantages of this method is that under certain conditions of regularity, the IFM estimator is consistent and asymptotically normal.

Also, in terms of numerical computation time, this method is better than the "direct" maximum likelihood method since it is simpler and faster.

2.5.2. Fit Test. To confirm whether the chosen parametric copula models the data well, it is necessary to perform a test. The most powerful tests are based on the processes $\sqrt{n}(\widehat{C} - C_{\widehat{\theta}})$, where $\widehat{C}$ and $C_{\widehat{\theta}}$ are, respectively, the empirical copula and the parametric copula.

The Cramer–von Mises statistic is by far the most used because it gives satisfactory results. It is defined by

$$\int_{[0;1]^d} n\left(\widehat{C} - C_{\widehat{\theta}}\right) \mathrm{d}\widehat{C}. \tag{17}$$

Other criteria such as the Akaike Criterion (AIC) and the Bayesian Inference Criterion (BIC) are very often used for the choice of the best copula. They are, respectively, defined by

$$\begin{aligned} \mathrm{AIC} &= 2m - \log\left(l(\theta)\right), \\ \mathrm{BIC} &= m^* \log(n) - \log\left(l(\theta)\right), \end{aligned} \tag{18}$$

where $l(\theta)$ is the model likelihood for the estimated parameter θ, m the number of estimated parameters, and n the data size.

3. Methodology and Application

The price of a multivariate option is a function of the density associated to the joint distribution. Thus, their valuation requires the determination of the joint risk-neutral density. To do this, it suffices to determine the marginal risk-neutral densities and then to choose the copula that best describes their dependence structure by using Sklar's theorem. This perspective is possible because the objective copula can be matched with the risk-neutral joint copula, under certain conditions (see Rosenbergh [20]).

3.1. Methodology. Our approach consists firstly in determining the marginal risk-neutral distributions by the procedure used by Idier et al. [15]. This is done in order to take into account the effect of extreme values in the margins. We will also limit ourselves to the case of a mixture of two Gaussians in this study. Clearly, we will use these two steps:

(1) Estimate the parameters of the mixture regimes.

(2) Estimate the parameters of the stochastic discount factor, using (31).

Then, we will choose among the families of copulas listed in the section the one that best suits the study. And finally, we will determine the prices of the multivariate options by numerical integration (Monte Carlo method) by using the formulas provided below for multivariate options considered. For doing so, and to complete the procedure, we will proceed by using these last four steps.

(3) Generate, by using our historical data returns, a sample $x_i^* \sim F_i^*$, $i = 1, \ldots, d$. Each risk-neutral marginal distribution $F_i^*, i = 1, \ldots, d$, has the density given by (4) with the adequate parameters, estimated in steps (1) and (2).

(4) Transform the vector sample $(x_1^*, \ldots, x_d^*)$ to vector of pseudo sample $(u_1^*, \ldots, u_d^*) \sim C_\theta^*$.

(5) Estimate the parameters θ of the copula C_θ^* using $(u_1^*, \ldots, u_d^*)$.

(6) Use Monte Carlo numerical integration method to calculate the option prices.

We will be particularly interested by rainbow options (those relating to the maximum or the minimum of several assets, etc.). These kinds of options have been the subject of many studies as in Stulz [21] and Jonshon [22].

Consider d assets whose price at maturity T is denoted by $S_1^T, \ldots, S_d^T$ and denote by $X_1^T, \ldots, X_d^T$, respectively, the returns associated to each asset at instant T (with for all $i = 1, \ldots, d$; $X_i^T = \ln(S_i^T / S_i^t)$). For a chosen strike price K, we consider the following different types of rainbows: spreads option; option on the maximum; option on the minimum and digital option.

3.2. Spread Option. Having a pay-off equal to $\max(S_2^T - S_1^T - K; 0)$, its value is calculated by

$$\begin{aligned} V_{\mathrm{OS}}(t) &= e^{-r(T-t)}\mathbb{E}^*\left\{\max\left(S_2^T - S_1^T - K; 0\right)\right\} = e^{-r(T-t)}\mathbb{E}^*\left\{\int_{\mathbb{R}} \mathbb{I}_{\{K+S_1^T \le x \le S_2^T\}}\mathrm{d}x\right\} \\ &= e^{-r(T-t)}\mathbb{E}^*\left\{\int_{\mathbb{R}} \mathbb{I}_{\{K+S_1^T \le x\}} - \mathbb{I}_{\{K+S_1^T \le x \, et \, S_2^T \le x\}}\mathrm{d}x\right\}, \end{aligned} \tag{19}$$

which gives

$$\begin{aligned} V_{\mathrm{OS}}(t) &= e^{-r(T-t)}\int_{-\infty}^{+\infty} \mathbb{P}^*\left\{K + S_1^T \le x\right\} - \mathbb{P}^*\left\{K + S_1^T \le x \, et \, S_2^T \le x\right\}\mathrm{d}x \\ &= e^{-r(T-t)}\int_{-\infty}^{+\infty} \mathbb{P}^*\left\{\frac{S_1^T}{S_1^t} \le \frac{x-K}{S_1^t}\right\} - \mathbb{P}^*\left\{\frac{S_1^T}{S_1^t} \le \frac{x-K}{S_1^t} \; et \; \frac{S_2^T}{S_2^t} \le \frac{x}{S_2^t}\right\}\mathrm{d}x, \end{aligned} \tag{20}$$

and finally

$$\begin{aligned} V_{\mathrm{OS}}(t) &= e^{-r(T-t)}\int_{-\infty}^{+\infty} \mathbb{P}^*\left\{X_1^T \le \log\left(\frac{x-K}{S_1^t}\right)\right\} - \mathbb{P}^*\left\{X_1^T \le \log\left(\frac{x-K}{S_1^t}\right) et \; X_2^T \le \log\left(\frac{x}{S_2^t}\right)\right\}\mathrm{d}x \\ &= e^{-r(T-t)}\int_K^{+\infty} e^{r(T-t)}\frac{\partial P_{t,1}}{\partial k_1}\left(T; \log(x_1 - k_1)\right) - \\ &\quad - C\left(e^{r(T-t)}\frac{\partial P_{t,1}}{\partial k_1}\left(T; \log(x_1 - k_1)\right); e^{r(T-t)}\frac{\partial P_{t,2}}{\partial k_2}\left(T; \log(x_2)\right)\right)\mathrm{d}x, \end{aligned} \tag{21}$$

where

$$\begin{aligned} x_i &= \frac{x}{S_i^t}; \\ k_i &= \frac{K}{S_i^t}, \end{aligned} \tag{22}$$

and $P_{t,i}$ is the put i price, for $i = 1, 2$.

3.3. Call on the Maximum. Its pay-off is equal to $\max\{\max(S_1^T, \ldots, S_d^T) - K; 0\}$. Thus, its price at maturity is given by

$$\begin{aligned} V_{\mathrm{CMax}}(t) &= e^{-r(T-t)}\mathbb{E}^*\left\{\max\left\{\max\left(S_1^T, \ldots, S_d^T\right) - K; 0\right\}\right\} = e^{-r(T-t)}\mathbb{E}^*\left\{\int_{\mathbb{R}} \mathbb{I}_{\{k \le x \le \max\left(S_1^T, \ldots, S_d^T\right)\}}\mathrm{d}x\right\} \\ &= e^{-r(T-t)}\mathbb{E}^*\left\{\int_K^{+\infty} 1 - \mathbb{I}_{\{\max\left(S_1^T, \ldots, S_d^T\right) \le x\}}\mathrm{d}x\right\}, \end{aligned} \tag{23}$$

we obtain

$$\begin{aligned} V_{\mathrm{CMax}}(t) &= e^{-r(T-t)}\int_K^{+\infty} 1 - \mathbb{P}^*\left\{S_1^T \le x, \ldots, S_d^T \le x\right\}\mathrm{d}x \\ &= e^{-r(T-t)}\int_K^{+\infty} 1 - \mathbb{P}^*\left\{X_1^T \le x_1, \ldots, X_d^T \le x_d\right\}\mathrm{d}x, \end{aligned} \tag{24}$$

and finally

$$V_{\mathrm{CMax}}(t) = e^{-r(T-t)}\int_K^{+\infty} 1 - C\left(e^{r(T-t)}\frac{\partial P_{t,1}}{\partial k_1}\left(T; x_1\right), \ldots, e^{r(T-t)}\frac{\partial P_{t,d}}{\partial k_d}\left(T; x_d\right)\right)\mathrm{d}x, \tag{25}$$

where $x_i = \log(x/S_i^t)$ and $P_{t,i}$ is the put i price, for $i = 1, 2$.

3.4. Call on the Minimum. It admits for pay-off $\max\{\min(S_1^T, \ldots, S_d^T) - K; 0\}$ and its value at maturity is then defined by

$$V_{\mathrm{CMin}}(t) = e^{-r(T-t)}\mathbb{E}^*\left\{\max\left\{\min\left(S_1^T, \ldots, S_d^T\right) - K; 0\right\}\right\} = e^{-r(T-t)}\mathbb{E}^*\left\{\int_{\mathbb{R}} \mathbb{I}_{\{k \le x \le \min(S_1^T, \ldots, S_d^T)\}} \mathrm{d}x\right\}, \tag{26}$$

which is equal to

$$V_{\mathrm{CMin}}(t) = e^{-r(T-t)}\mathbb{E}^*\left\{\int_K^{+\infty} \mathbb{I}_{\{\min(S_1^T, \ldots, S_d^T) \ge x\}} \mathrm{d}x\right\} = e^{-r(T-t)} \int_K^{+\infty} \mathbb{P}^*\left\{S_1^T \ge x, \ldots, S_d^T \ge x\right\} \mathrm{d}x, \tag{27}$$

and at the end, we obtain

$$\begin{aligned} V_{\mathrm{CMin}}(t) &= e^{-r(T-t)} \int_K^{+\infty} \mathbb{P}^*\left\{X_1^T \ge x_1, \ldots, X_d^T \ge x_d\right\} \mathrm{d}x \\ &= e^{-r(T-t)} \int_K^{+\infty} \overline{C}\left(-e^{r(T-t)} \frac{\partial C_{t,1}}{\partial k_1}(T; x_1), \ldots, -e^{r(T-t)} \frac{\partial C_{t,d}}{\partial k_d}(T; x_d)\right) \mathrm{d}x. \end{aligned} \tag{28}$$

where $x_i = \log(x/S_i^t)$ and $C_{t,i}$ is the call i price, for $i = 1, 2$.

3.5. Digital Option. It has for pay-off $\mathbb{I}_{\{S_1^T \ge K_1, \ldots, S_d^T \ge K_d\}}$. Thus, its value at maturity is given by

$$V_{\mathrm{ODig}}(t) = e^{-r(T-t)}\mathbb{E}^*\left\{\mathbb{I}_{\{S_1^T \ge K_1, \ldots, S_d^T \ge K_d\}}\right\} = e^{-r(T-t)}\mathbb{P}^*\left\{S_1^T \ge K_1, \ldots, S_d^T \ge K_d\right\}, \tag{29}$$

which gives us

$$V_{\mathrm{ODig}}(t) = e^{-r(T-t)}\overline{C}\left(-e^{r(T-t)} \frac{\partial C_{t,1}}{\partial k_1}(T; k_1), \ldots, -e^{r(T-t)} \frac{\partial C_{t,d}}{\partial k_d}(T; k_d)\right), \tag{30}$$

where $k_i = K_i/S_i^t$; $C_{t,i}$ is the call i price, for $i = 1, \ldots, d$, and $\overline{C}$ the survival copula.

Remark 3. Not to forget that the quantities $-e^{r(T-t)}(\partial C_{t,d}/\partial k)(T; k)$ and $e^{r(T-t)}(\partial P_{t,d}/\partial k)(T; k)$ are both equal to the risk-neutral distribution whose density is given by relation (4) for our study.

3.6. Applications. We will focus on the bivariate options on the pair of *Atos* and *Dassault Systems* shares. The data were obtained from *Investing.com* and relate to the components of the *CAC40 index* of the Paris stock exchange. The collected data concern the closing prices for the period from July 01, 2014, to June 30, 2020 (1534 days).

At first, for each of the two assets (Atos; Dassault Systems), the parameters of the two Gaussian regimes constituting the Gaussian mixture are determined (Table 1) as well as the proportions of each diet.

In the next step, we determine the parameters $(\alpha_t; \beta_t)$ of the stochastic discount factor defined by relation (3) thanks to the assumptions of the model, in particular that of lack of arbitration opportunity. These parameters are determined as unique solution of (see [15])

$$\begin{cases} \exp(r_{t+1} + \beta_t)\mathbb{E}\{\exp(\alpha_t X_{t+1})\} = 1, \\ \exp(\beta_t)\mathbb{E}\{\exp((\alpha_t + 1)X_{t+1})\} = 1, \end{cases} \tag{31}$$

where r_{t+1} is the risk-free rate at $t + 1$, known at t.

The results obtained for a risk-free rate $r_{t+1} = 0.025$ are presented in Table 2.

Table 1: Estimated parameters of the Gaussian mixture and their proportion.

		Regime 1	Regime 2	Gaussian mixture	Empirical distribution
Atos	Moyenne	−0.0072328	0.000764489	0.00013704	0.0001370
	Ecart-type	0.0603574	0.013530408	0.02142835	0.0214349
	Skewness	0	0	−0.6131518	−2.823688
	Kurtosis	3	3	15.7	42.81632
	Proportion	0.07845771	0.921542310	–	–
Dassault systems	Moyenne	0.00110506	−0.00101651	0.0007598786	0.0007599
	Ecart-type	0.01014017	0.03315738	0.01577694	0.01630201
	Skewness	0	0	−0.03745108	0.1949504
	Kurtosis	3	3	10.00661	12.39682
	Proportion	0.83729906	0.16270094	–	–

Table 2: Stochastic discount factor parameters for data returns with a risk-free rate $r_{t+1} = 0.025$.

	Atos	Dassault
α_t	36.1209027	37.70565
β_t	−0.3610132	−0.3500945

3.7. Copulas Fitting Results. We can now fit the copulas to our data because all the parameters needed for express the risk-neutral density (given by (6)) are known. We present below the results of the copula estimates associated with our data. In each case, we will base ourselves on the Cramer–von Mises statistic and/or the AIC criteria for the choice of the best copula.

In Table 3, we present the estimated parameters (and their Cramer–von Mises statistics) for bivariate copulas. It then emerges that the three copulas with the best Cramer–von Mises statistic are Tawn's copula, Frank's copula, and Gumbel's copula in that order.

Table 4 gives the AIC and BIC of the parameters estimated for the bivariate copulas chosen. Based on these criteria, the four best candidate copulas for our bivariate data are the normal copula, the Husler–Reiss's copula, the Galambos's copula, and the Gumbel's copula.

We can notice that, for our data, the performance of each fitted copula differs according to the criterion. It is then difficult to make a particular choice on a copula in such situation on the basis of the two criteria (Cramer–von Mises statistic vs AIC) combined. Nevertheless, if there is a choice to be made between these two criteria, it would be more judicious to base oneself on the Cramer–von Mises test.

3.8. Options Prices by Monte Carlo Approach. In this section, we give the simulation results of the prices (for one period $T = t + 1$) of all options presented in the section above based on the basket (Atos, Dassault systems). Tables 5 and 6 give, respectively, the prices of the call on maximum and the call of maximum. We fix the price of each asset of the bivariate basket to 120. The values of their prices are calculated when it is out-of-the money (OTM), at-the-money (ATM), and in-the-money (ITM). For the cases of digital option and the spread option, we fix, respectively, the price of the basket to $S = (120, 130)$ and $S = (100, 120)$. We give the prices of these options in Tables 7 and 8 obtained also for different strikes. By using the formulas in Section 2, the prices are calculated by Monte Carlo numerical expectation calculation method with $N = 10^5$ simulations (the choice of the bivariate options with our underlying is simply for academic interest. In fact, they are not exchanged on the market). Indeed, since most of these formulas are expressed in terms of integrals, we can transform each of them into an expectation of a random variable with suitably chosen distribution. In our case, we make first a change of the integral bounds $(K, +\infty)$ to $(0, +\infty)$ and choose the Paréto distribution.

For the case of the call on maximum (Table 5), the price obtained by normal is superior to the prices with all others copulas (Archimedean and extreme) in all the three situations without only the case when it is at-the-money with Cayton's copula. We notice that the prices obtained with the others are approximately the same (weak discrepancies). The normal copula overestimates the price compared to Tawn's copula which has the best fitness test.

In the case of the call on minimum (Table 6), the normal copula presents a price which is lower than that obtained by any extreme value copula when it is at-the-money or out-of-the money. We notice the contrary when it is in-the-money.

Particularly, the normal copula gives a highest price than Tawn's copula when it is in-the-money with a small discrepancy. And, when it is at-the-money or out-of-the money Tawn's copula produces a price superior to that of normal copula with a fairly larger gap than the first situation.

For prices of the digital option (Table 7), that obtained by normal copula is inferior to the others calculated by extreme copula in the three situations of valuation.

For the spreads option (Table 8), we notice that when it is at-the-money the normal copula gives the highest price and Tawn's copula gives the smallest price. Otherwise, when it is in-the-money the normal copula gives the smallest price. Finally, when the option is out-of-the money, the normal copula gives the second great price. Comparatively to the price obtained with Tawn's copula, the price calculated with normal copula is the greatest when the option is at-the-money and out-of-the-money but the smallest when it is in-the-money see Abba-Mallam et al. [23, 24].

Remark 4. When X and Y are two random variables modeling the returns of two shares, having an extreme value copula, one can compute the discordance function for more information about the dependence between these variables. For more details on this measure, see Dossou-Gbété et al. [25].

Table 3: Parameters of bivariate copulas selected and their Cramer–von Mises test statistics.

	Normal	Clayton	Gumbel	Frank	Tawn	Galambos	Husler–Reiss
Parameter	0.2822	0.1766	1.344	2.3166	0.6868	0.5995	0.9798
Statistic of test	1.469	3.257	0.72485	0.636	0.6136	0.79162	0.84238
p value	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005

Table 4: AIC and BIC of the estimated parameters (by the maximum likelihood) for the selected bivariate copulas.

	Normal	Clayton	Gumbel	Frank	Tawn	Galambos	Husler–Reiss
$\log(l(\theta))$	365.9	325.1	345.4	291.8	320	354.7	358
AIC	−729.8	−648.2	−688.8	−581.6	−638	−707.4	−714
BIC	−724.46	−642.86	−683.46	−576.26	−632.66	−702.06	−708.66

Table 5: Prices of the call on maximum with different strikes.

	Normal	Clayton	Gumbel	Frank	Galambos	Tawn	Husler–Reiss
OTM ($K = 130$)	2.7126	2.6947	2.6642	2.6888	2.6593	2.6664	2.6617
ATM ($K = 120$)	7.328	7.417	7.0226	7.101	7.0216	7.0264	7.0245
ITM ($K = 110$)	17.211	17.057	16.743	16.794	16.674	16.660	16.673

Table 6: Prices of the call on minimum with different strikes K.

	Normal	Clayton	Gumbel	Frank	Galambos	Tawn	Husler–Reiss
OTM ($K = 130$)	0.0386	0.02188	0.05882	0.03639	0.06085	0.05437	0.06224
ATM ($K = 120$)	1.181	0.995	1.361	1.268	1.356	1.369	1.353
ITM ($K = 110$)	10.628	10.183	10.559	10.451	10.536	10.523	10.543

Table 7: Prices of the bivariate digital option (paying one unit of the money) with different strikes K.

	Normal	Clayton	Gumbel	Frank	Galambos	Tawn	Husler–Reiss
OTM	0.02283713	0.014745475	0.03258418	0.02359763	0.03306132	0.03181797	0.03329281
ATM	0.5208968	0.507406	0.5273577	0.5344094	0.5258449	0.53006026	0.5247992
ITM	0.97519641	0.9751979	0.9751963	0.9751963	0.9751963	0.9751964	0.9751963

Table 8: Prices of the bivariate spread option with different strikes K.

	Normal	Clayton	Gumbel	Frank	Galambos	Tawn
OTM ($K = 30$)	0.04626	0.05037	0.01244	0.02615	0.00898	0.01909
ATM ($K = 20$)	2.1429	1.379	0.9609	1.0818	1.006	0.8878
ITM ($K = 10$)	6.5442	7.8945	7.4151	7.8331	7.5363	7.4094

4. Conclusion and Discussions

This paper proposes an approach that allows taking into account the effect of extreme values in the marginal and joint distribution of the underlying for the valorisation of multivariate options. For doing so, at first, each marginal distribution of the returns of any underlying asset is modelled by a mixture of Gaussian as in Idier et al. [15] and the dependence structure is modelled by a copula. The choice of the best copula is confirmed by fitting and goodness test fit.An application is made on the basket (Atos, Dassault systems) of the financial market CAC40 which reveals that Tawn's copula is the best for modeling the dependence structure of their returns. Thus, the prices of four type of options are calculated by use of Monte Carlo simulation. The simulations results show that the normal copula overestimates the prices for the call on maximum and the spread option when they are at-the money. In the case of digital and

call on minimum, this copula underestimates the prices when the options are at-the money.

Data Availability

Data are available on request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this study.

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