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Asymptotic behavior of neutral stochastic partial functional integro-differential equations driven by a fractional Brownian motion

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Abstract

This paper deals with the existence, uniqueness and asymptotic behavior of mild solutions to neutral stochastic delay functional integro-differential equations perturbed by a fractional Brownian motion B^H, with Hurst parameter $H \in (\frac{1}{2}, 1)$. The main tools for the existence of solution is a fixed point theorem and the theory of resolvent operators developed in Grimmer [R. Grimmer, Trans. Amer. Math. Soc., **273** (1982), 333–349.], while a Gronwall-type lemma plays the key role for the asymptotic behavior. An example is provided to illustrate the results of this work.

Keywords: Resolvent operators, C_0-semigroup, Wiener process, Mild solutions, Fractional Brownian motion, Exponential decay of solutions.
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1. Introduction

In the paper [5] we proved the existence and uniqueness of mild solution for the following abstract stochastic neutral partial functional integro-differential equations

$$\begin{cases} d\left[u(t)+G(t,u(t-r(t)))\right]=A\left[u(t)+G(t,u(t-r(t)))\right]dt \\ \qquad +\left[\int_0^t B(t-s)[u(s)+G(s,u(s-r(s)))]ds+F(t,u(t-\delta(t)))\right]dt \\ \qquad +\sigma(t)dB^H(t) \quad \text{for } t\in[0,T], \\ u_0=\varphi, \quad \text{i.e., } u(t)=\varphi(t), \quad -r\le t\le 0, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X with domain $D(A)$, $B(t)$ is a closed linear operator on X with domain $D(B)\supset D(A)$ which is independent of t, B^H is a fractional Brownian motion on a real and separable Hilbert space Y. The functions $r,\delta:[0,+\infty)\to[0,\tau](\tau>0)$ are continuous, and $G,F:[0,+\infty)\times X\to X, \sigma:[0,+\infty)\to\mathcal{L}_2^0(Y,X)$ are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see Section 2).
Now we are interested in analyzing the long-term behavior of the mild solutions to our problem (1.1). In particular, the exponential decay of solutions to zero in mean square. However, before carrying out such study, we will also improve result about existence and uniqueness of mild solution in [5] in the sense that the nonlinear term $F(\cdot,\cdot)$, which contained a continuous variable delay, will be allowed here to be of functional form $f(t,x_t)$, i.e., depending of the segment solution $x_t:[-\tau,0]\to X$ given by $x_t(s)=x(t+s)$ for $s\in[-\tau,0]$. With this general formulation we can also relax the continuity assumption on the delay function δ to simply measurability. But other types of delay, such a distributed bounded delay can be also included in this set-up (see, for instance [7, Remark 2.1] for more details). Consequently, our model to be analyzed is

$$\begin{cases} d\left[u(t)+G(t,u(t-r(t)))\right]=A\left[u(t)+G(t,u(t-r(t)))\right]dt \\ \qquad +\left[\int_0^t B(t-s)[u(s)+G(s,u(s-r(s)))]ds+f(t,u_t)\right]dt \\ \qquad +\sigma(t)dB^H(t) \quad \text{for } t\in[0,T], \\ u_0=\varphi, \quad \text{i.e., } u(t)=\varphi(t), \quad -r\le t\le 0, \end{cases} \tag{1.2}$$

These equations provide useful and important mathematical models for engineering problems and for this reason have received much attention in recent years (see, e.g. [2],[11],[16],[18], [4],[13] and references therein). The literature related to neutral differential equations of the type (1.2) is not vast, and this is why we aim to continue the research initiated in [5]. Thus, in Section 2, we introduce some notations, concepts concerning resolvent operators, basic results about fractional Brownian motion and the Wiener integral over Hilbert space. In Section 3, we establish the existence and uniqueness of mild solutions for the system (1.2) by using a fixed point theorem as a main tool. Section 4 is devoted to prove our stability results with the help of a Gronwall-like lemma proved in [9] and which has revealed as a helpful tool in proving stability results for stochastic differential equations with delays. Finally, in Section 5 we will exhibit an example to illustrate our previous abstract results.

2. Wiener Process and deterministic integro-differential equations

2.1. Wiener process

In this section we introduce the fractional Brownian motion as well as the Wiener integral with respect to it. We also establish some important results which will be needed throughout the paper.
Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space.

Definition 2.1. Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H(t),\ t \in \mathbb{R}$, with covariance function

$$R_H(s,t) = \mathbb{E}\big[\beta^H(t)\beta^H(s)\big] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),\ \ t,s \in \mathbb{R}$$

is called a *two-sided one-dimensional fractional Brownian motion (fBm), and* H *is the Hurst parameter.*

Now we aim at introducing the Wiener integral with respect to the one-dimensional fBm β^H. Let $T > 0$ and denote by Λ the linear space of $\mathbb{R}$-valued step function on $[0,T]$, that is $\phi \in \Lambda$ if

$$\phi(t) = \sum_{i=1}^{n-1} x_i 1_{[t_i,t_{i+1})}(t),$$

where $t \in [0,T],\ x_i \in \mathbb{R}$ and $0 = t_1 < t_2 < \cdots < t_n = T$. For $\phi \in \Lambda$ we define its Wiener integral with respect to β^H as

$$\int_0^T \phi(s)d\beta^H(s) = \sum_{i=1}^{n-1} x_i(\beta^H(t_{i+1}) - \beta^H(t_i)).$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of Λ with respect to the scalar product $\langle 1_{[0;t]}, 1_{[0;s]}\rangle_{\mathcal{H}} = R_H(t,s)$.

Then the mapping

$$\phi = \sum_{i=1}^{n-1} x_i 1_{[t_i,t_{i+1})} \ \to \int_0^T \phi(s)d\beta^H(s)$$

is an isometry between Λ and the linear space span$\{\beta^H,\ \ t \in [0,T]\}$, which can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fBm $\overline{\text{span}}^{L^2(\Omega)}\{\beta^H,\ \ t \in [0,T]\}$ (see [17]). The image of an element $\varphi \in \mathcal{H}$ by this isometry is called the Wiener integral of φ with respect to β^H. Our next goal is to give an explicit expression of this integral. To this end, consider the Kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

where $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$, with B denoting the Beta function and $t \leq s$. It is not difficult to see that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H(\frac{t}{s})^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}.$$

Consider the linear operator $K_H^* : \Lambda \longrightarrow L^2([0,T])$ given by

$$(K_H^*\varphi)(s) = \int_s^t \varphi(t)\frac{\partial K}{\partial t}(t,s)dt.$$

Then

$$K_H^* 1_{[0;t]}\ (s) = K_H(t,s)1_{[0;t]}\ (s)$$

and K_H^* is an isometry between Λ and $L^2([0,T])$ that can be extended to Λ (see [1]).

Considering $W = \{W(t),\ \ t \in [0,T]\}$ defined by

$$W(t)\ =\ \beta^H((K_H^*)^{-1}1_{[0;t]}),$$

it turns out that W is a Wiener process and β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t\ K_H(t,s)dW(s).$$

In addition, for any $\varphi \in \Lambda$,

$$\int_0^T \varphi(s)d\beta^H(s) = \int_0^T (K_H^*\varphi)(t)dW(t)$$

if and only if $K_H^*\varphi \in L^2([0,T])$.

Also denoting $L^2_{\mathcal{H}}([0,T]) = \{\varphi \in \Lambda,\ K_H^*\varphi \in L^2([0,T])\}$, since $H > \frac{1}{2}$, we have

$$L^{\frac{1}{H}}([0,T]) \subset L^2_{\mathcal{H}}([0,T]), \tag{2.1}$$

see [14]. Moreover, the following useful result holds:

Lemma 2.2. *(Nualart [15]) For* $\varphi \in L^{\frac{1}{H}}([0,T])$,

$$H(2H-1)\int_0^T\int_0^T |\varphi(r)||\varphi(u))||r-u|^{2H-2}drdu \leq c_H\|\varphi\|^2_{L^{\frac{1}{H}}([0,T])}.$$

Next we are interested in considering a fBm with values in a Hilbert space and giving the Definition of the corresponding stochastic integral.

Let $(X, \|.\|_X, (.,.)_X)$ and $(Y, \|.\|_Y, (.,.)_Y)$ be separable Hilbert spaces. Let $\mathcal{L}(Y,X)$ denote the space of all bounded linear operator from Y to X. Let $Q \in \mathcal{L}(Y,X)$ be a non-negative self-adjoint operator . Denote by $L^0_Q(Y,X)$ the space of $\vartheta \in L(Y,X)$ such that $\vartheta Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is given by

$$|\vartheta|^2_{L^0_Q(Y,X)} = \left|\vartheta Q^{\frac{1}{2}}\right|_{HS} = tr(\vartheta Q \vartheta^*).$$

Then ϑ is called a Q-Hilbert-Schmidt operator from Y to X.

Let $\{\beta_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. When one considers the following series

$$\sum_{n=1}^{\infty} \beta_n^H(t)e_n, \quad t \geq 0,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal bais in X, this series does not necessarily converge in the space Y. Thus we consider a Y-valued stochastic process

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t)Q^{\frac{1}{2}}e_n, \quad t \geq 0.$$

If Q is a non-negative self-adjoint trace class operator, this series converges in the space Y, that is, it holds that $B_Q^H(t) \in L^2(\Omega, Y)$. Then, we say that the above $B_Q^H(t)$ is a Y-valued Q-cylindrical fractional Brownian motion with covariance operator Q. For example, if $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \sigma_n e_n$, assuming that Q is a nuclear operator in Y (that is, $\sum_{n=1}^{\infty}\sigma_n < \infty$), then the stochastic process

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t)Q^{\frac{1}{2}}e_n = \sum_{n=1}^{\infty} \sqrt{\sigma_n}\beta_n^H(t)e_n, \quad t \geq 0,$$

is well-defined as a Y-valued Q-cylindrical fractional Brownian motion.

Let $\varphi : [0,T] \to L^0_Q(Y,X)$ such that

$$\sum_{n=1}^{\infty} \left\|K_H^*(\varphi Q^{\frac{1}{2}}e_n)\right\|_{\mathcal{L}^2([0,T];X)} < \infty. \tag{2.2}$$

Definition 2.3. Let $\varphi : [0,T] \to L_H^0(Y,X)$ satisfy (2.2). Then, its stochastic integral with respect to the fBm B_Q^H is defined, for $t \geq 0$, as follows

$$\int_0^t \varphi(s)dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s)Q^{\frac{1}{2}}e_n\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t (K_H^*(\varphi Q^{\frac{1}{2}}e_n))(s)dW(s). \tag{2.3}$$

Notice that if

$$\sum_{n=1}^{\infty} \|\varphi Q^{\frac{1}{2}}e_n\|_{L^{\frac{1}{H}}([0,T];X)} < \infty,$$

then in particular (2.2) holds, which follows immediately from (2.1).

Now we end this subsection by stating the following result which is fundamental to prove our result. It can proved by similar arguments as those used to prove Lemma 2 in Caraballo et al., [6].

Lemma 2.4. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi\|_{\mathcal{L}_2^0}^2 ds < \infty$ then the above sum in (2.3) is well defined as a X-valued random variable and we have*

$$E\left\|\int_0^t \psi(s)dB^H(s)\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|_{\mathcal{L}_2^0}^2 ds.$$

Proof. See [3].

2.2. Partial integro-differential equations in Banach spaces

In this section, we recall some fundamental results needed to establish our results. Regarding the theory of resolvent operators we refer the reader to [12]. Throughout the paper, X is a Banach space, A and $B(t)$ are closed linear operators on X. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$\|y\|_Y := \|Ay\|_X + \|y\|_X \quad \text{for } y \in Y.$$

The notations $C([0,+\infty);Y), \mathcal{B}(Y,X)$ stand for the space of all continuous functions from $[0,+\infty)$ into Y, the set of all bounded linear operators from Y into X, respectively. We consider the following Cauchy problem

$$\begin{cases} v'(t) &= Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds, \quad \text{for } t \geq 0, \\ v(0) &= v_0 \in X. \end{cases} \tag{2.4}$$

Definition 2.5. ([12]). A resolvent operator for Eq. (2.4) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$, satisfying the following properties:

(i) $R(0) = I$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.

(iii) $R(t) \in \mathcal{L}(Y)$ for $t \geq 0$. For $x \in Y$, $R(\cdot)x \in C^1([0,+\infty);X) \cap C([0,+\infty);Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad \text{for } t \geq 0. \end{aligned}$$

The resolvent operators play an important role to study the existence of solutions and to give a variation of constants formula for nonlinear systems. We need to know when the linear system (2.4) has a resolvent operator. For more details on resolvent operators, we refer the reader to [12]. The following Theorem gives a satisfactory answer to this problem and it will be used in this work to develop our main results.

In what follows we suppose the following assumptions :

(H1) A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X.

(H2) For all $t \geq 0$, $B(t)$ is a closed linear operator from $D(A)$ to X, and $B(t) \in \mathcal{B}(Y, X)$. For any $y \in Y$, the map $t \to B(t)y$ is bounded, differentiable and the derivative $t \to B'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.6. *([12, Theorem 3.7]) Assume that* ***(H1)-(H2)*** *hold. Then there exists a unique resolvent operator for the Cauchy problem (2.4).*

In what follows, we establish some results for the existence of solutions of the following integro-differential equation

$$\begin{cases} v'(t) &= Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds + q(t), \quad \text{for} \quad t \geq 0, \\ v(0) &= v_0 \in X, \end{cases} \tag{2.5}$$

where $q : [0, +\infty[\to X$ is a continuous function.

Definition 2.7. ([12]). A continuous function $v : [0, +\infty) \to X$ is said to be a strict solution of Eq. (2.5) if
(i)$v \in C^1([0, +\infty); X) \cap C([0, +\infty); Y)$,
(ii) v satisfies Eq. (2.5) for $t \geq 0$.

Remark 2.8. From this Definition, we deduce that $v(t) \in D(A)$, and the function $B(t-s)v(s)$ is integrable, for all $t \geq 0$ and $s \in [0, t]$.

Theorem 2.9. *([12, Theorem 2.5]). Assume that* ***(H1)-(H2)*** *hold . If v is a strict solution of Eq. (2.5), then*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \text{for} \quad t \geq 0. \tag{2.6}$$

Accordingly, we have the following Definition.

Definition 2.10. ([12]). A function $v : [0, +\infty) \to X$ is called a mild solution of (2.5) if v satisfies the variation of constants formula (2.6), for $v_0 \in X$.

The next Theorem provides sufficient conditions for the regularity of solutions of Eq. (2.5). Namely we establish a sufficient condition ensuring when a mild solution is a strict one.

Theorem 2.11. *([12, Corollary 3.8]) Let $q \in C^1([0, +\infty); X)$ and v be defined by (2.6). If $v_0 \in D(A)$, then v is a strict solution of Eq. (2.5).*

3. Existence of mild solutions for Eq (1.1)

In this section, we establish the existence and uniqueness of mild solutions of Eq. (1.2) under Lipschitz conditions. We use the following hypotheses to prove our results.

(H3) $f : [0, T] \times C([-\tau, 0]; X) \to X$ is a family of non-linear operators defined for almost every t (a.e. t) which satisfies

$(f.1)$ The mapping $t \in (0,T) \to f(t,\xi) \in X$ is Lebesgue measurable, for a.e. t and for all $\xi \in C([-\tau,0];X)$.
$(f.2)$ There exists a constant $c_L > 0$ such that for all $\phi, \psi \in C([-\tau,0];X)$

$$\|f(t,\phi) - f(t,\psi)\|_X^2 \leq c_L \|\phi - \psi\|^2_{C([-\tau,0];X)},$$

$(f.3)$ There exists a constant $c_f > 0$ such that for any $x, y \in C([-\tau,T];X)$ and $t \in [0,T]$,

$$\int_0^t \|f(s,x_s) - f(s,y_s)\|_X^2 \, ds \leq c_f \int_{-\tau}^t \|x(s) - y(s)\|_X^2 \, ds.$$

$(f.4)$

$$\int_0^T \|f(s,0)\|_X^2 ds < \infty.$$

(H4) The function $G\ : [0,+\infty[\times X \to X$ satisfies the following conditions: there exist positive constant $c_3, c_4, 0 < c_3 < 1$ such that, for all $t \in [0,T]$ and $x, y \in X$

$$\|G(t,x) - G(t,y)\|_X \leq c_3 \|x - y\|_X,$$

$$\|G(t,x)\|_X^2 \leq c_4(1 + \|x\|_X^2).$$

(H5) The function G is continuous in the quadratic mean sense:
For all $x \in C([0,T], L^2(\Omega,X))$, $\quad \lim_{t\to s} \mathbb{E}\,\|G(t,x(t)) - G(s,x(s))\|_X^2 = 0$.
(H6) The function $\sigma\ :\ [0,+\infty[\to \mathcal{L}_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0} \, ds < \infty, \ \ \forall T > 0.$$

Remark 3.1. Observe that if f is defined for each $\phi \in C([-\tau,0];X)$ as $f(t,\phi) := F(t,\phi(-\delta(t)))$, where $F\ : [0,+\infty) \times X \to X$ satisfies **(H3)'** below, and $\delta(\cdot) : \mathbb{R} \to [0,\tau]$ is continuously differentiable with $\delta'(t) \leq \delta^* < 1$ for all $t \in \mathbb{R}$, then it holds (H3).
(H3)' The function $F : [0,+\infty[\times X \to X$ satisfies the following conditions: there exist positive constant c_1, c_2 such that, for all $t \in [0,T]$ and $x, y \in X$,

$$\|F(t,x) - F(t,y)\|_X \leq c_1 \|x - y\|_X,$$

$$\|F(t,x)\|_X^2 \leq c_2(1 + \|x\|_X^2).$$

Moreover, we assume that $\varphi \in C([-\tau,0], L^2(\Omega,X))$.
We now introduce the concept of mild solution of the Eq.(1.2).

Definition 3.2. An X-valued process $\{u(t),\ t \in [-\tau,T]\}$, is called a mild solution of Eq.(1.2) if $u \in C([-\tau,0], L^2(\Omega,X)), u(t) = \varphi(t)$ for $t \in [-\tau,0]$, and, for $t \in [0,T]$, satisfies

$$\begin{aligned} u(t) + G(t,u(t-r(t))) &= R(t)\left[\varphi(0) - G(0,\varphi(-r(0)))\right] + \textstyle\int_0^t R(t-s)f(s,u_s)ds \\ &\quad + \textstyle\int_0^t R(t-s)\sigma(s)dB^H(s) \quad \mathbb{P}-\text{a.s.} \end{aligned} \tag{3.1}$$

To show our main results we first recall the following Lemma.

Lemma 3.3. *(Caraballo et al. [8, Lemma 1]). For $z, z' \in X$ and $0 < c < 1$,*

$$\|z\|_X^2 \leq \frac{1}{1-c}\left\|z - z'\right\|_X^2 + \frac{1}{c}\left\|z'\right\|_X^2.$$

Theorem 3.4. *([5, theorem 3.3]) Under the assumptions **(H1)-(H6)**, for every $\varphi \in C([-\tau, T], L^2(\Omega, X))$ there exists a unique mild solution u to Eq.(1.1).*

Proof. As the proof parallels the steps in [5] we only emphasize the computations concerning the term f, and the reader is referred to the paper [5] for more details on the other terms.
For the given $T > 0$ and the initial function φ, we consider $C_T := C([-\tau, T], L^2(\Omega, X))$, the Banach space of all continuous functions from $[-\tau, T]$ into $L^2(\Omega, X)$ equipped with the supremum norm $\|\zeta\|_{C_T} = \sup_{z\in[-\tau,T]}(\mathbb{E}\,\|\zeta(z)\|^2)^{1/2}$, and define the set

$$S_T(\varphi) := \{u \in C([-\tau, 0], L^2(\Omega, X)) : u(s) = \varphi(s), \text{ for } \ s \in [-\tau, 0]\}.$$

$S_T(\varphi)$ is a closed subset of C_T, it is a complete metric space for the distance induced by the norm $\|.\|_{C_T}$. Then we define the operator Γ on $S_T(\varphi)$ by $\Gamma(u)(t) = \varphi(t)$ for $t \in [-\tau, 0]$, and for $t \in [0, T]$

$$\begin{aligned} \Gamma(u)(t) &= R(t)\,[\varphi(0) - G(0, \varphi(-r(0)))] - G(t, u(t - r(t))) \\ &\quad + \textstyle\int_0^t R(t-s) f(s, u_s) ds + \int_0^t R(t-s)\sigma(s) dB^H(s). \end{aligned} \tag{3.2}$$

Then, in order to prove the existence and uniqueness of mild solutions to Eq. (1.2) one has to find a fixed point for the operator Γ.
To this end we split the proof into two steps.

Step 1: For arbitrary $u \in S_T(\varphi)$, we first prove that $t \to \Gamma(u)(t)$ is continuous on the interval $[0, T]$ in the $L^2(\Omega, X)$-sense. Let $0 < t < T$ and $|h|$ be sufficiently small. Then, for any fixed $u \in S_T(\varphi)$, we have

$$\begin{aligned} &\|(\Gamma(u)(t+h) - (\Gamma(u)(t)\|_X \\ &\quad \le \|(R(t+h) - R(t))\,[\varphi(0) - G(0, \varphi(-r(0)))]\|_X \\ &\qquad + \|G(t+h, u(t+h-r(t+h))) - G(t, u(t-r(t)))\|_X \\ &\qquad + \left\| \int_0^{t+h} R(t+h-s) f(s, u_s) ds - \int_0^t R(t-s) f(s, u_s) ds \right\|_X \\ &\qquad + \left\| \int_0^{t+h} R(t+h-s)\sigma(s) dB^H(s) - \int_0^t R(t-s)\sigma(s) dB^H(s) \right\|_X \\ &\quad = \sum_{1\le i\le 4} I_i(h). \end{aligned}$$

Then terms $I_1(h), I_2(h)$ and $I_4(h)$ are analyzed as in [5]. As for $I_3(h)$, we will argue by assuming that $h > 0$ (similar estimates hold when $h < 0$). Then

$$\begin{aligned} I_3(h) &\le \left\| \int_0^t (R(t+h-s) - R(t-s)) f(s, u_s) ds \right\|_X + \left\| \int_t^{t+h} R(t+h-s) f(s, u_s) ds \right\|_X \\ &\le I_{31}(h) + I_{32}(h). \end{aligned}$$

Thanks to Hölder's inequality,

$$\mathbb{E}|I_{31}(h)|^2 \le t\mathbb{E} \int_0^t \|(R(t+h-s) - R(t-s)) f(s, u_s)\|_X^2\, ds.$$

Again exploiting properties (i) and (ii) of Definition 2.5, and (f.3)-(f.4), we have for each $s \in [0, t]$,

$$\lim_{h\to 0}(R(t+h-s) - R(t-s)) f(s, u_s) = 0,$$

and

$$\|(R(t+h-s) - R(t-s)) f(s, u_s)\|_X^2 \le \tilde{N}\, \|f(s, u_s)\|_X^2 \in \mathbb{L}^2([0, t] \times \Omega),$$

where $\tilde{N} = [2N^2 e^{2\beta(t+h)} + 2N^2 e^{2\beta t}]$. Then, by the Lebesgue Majorant Theorem, we conclude that

$$\lim_{h\to 0} \mathbb{E}|I_{31}(h)|^2 = 0.$$

Next, using property (ii) of Definition 2.5, condition (f.3) and Hölder's inequality, it follows

$$\begin{aligned}
\mathbb{E}|I_{32}(h)|^2 &\leq \mathbb{E}\left(\int_t^{t+h} \|R(t+h-s)\|\|f(s,u_s)\|_X ds\right)^2 \\
&\leq N^2 e^{2\beta T}\mathbb{E}\int_t^{t+h} \|f(s,u_s)\|_X^2 ds \\
&\leq hN^2 e^{2\beta T}\mathbb{E}\left(\int_t^{t+h} \|f(s,u_s)\|^2 ds\right)^2 \\
&\leq hN^2 e^{2\beta T}\mathbb{E}\int_0^T \|f(s,u_s) - f(s,0) + f(s,0)\|_X^2 ds \\
&\leq 2hN^2 e^{2\beta T}\mathbb{E}\int_0^T (\|f(s,u_s) - f(s,0)\|_X^2 + \|f(s,0)\|_X^2) ds \\
&\leq 2hN^2 e^{2\beta T}\left(\int_{-\tau}^T c_f \mathbb{E}\|u(s)\|_X^2 ds + \int_0^T \|f(s,0)\|_X^2 ds\right),
\end{aligned}$$

and then

$$\lim_{h\to 0} \mathbb{E}|I_{32}(h)|^2 = 0.$$

Hence, we can conclude that the function $t \to \Gamma(u)(t)$ is continuous on $[0,T]$ in the L^2-sense.

Step 2: Now we show that Γ is a contracting mapping in $S_{T_1}(\varphi)$ for some small enough $T_1 < T$.
For every $u,\ v \in S_T(\varphi)$ and $t \in [0,T]$, by using Lemma 3.3 we obtain

$$\begin{aligned}
\|\Gamma(u)(t) - \Gamma(v)(t)\|_X^2 &\leq \frac{1}{c_3}\|G(t,u(t-r(t))) - G(t,v(t-r(t)))\|_X^2 \\
&\quad + \frac{1}{1-c_3}\left\|\int_0^t R(t-s)(f(s,u_s) - f(s,v_s)) ds\right\|_X^2 .
\end{aligned}$$

Owing to the Lipschitz properties of F and G, combined with Hölder's inequality and (f.3), we obtain

$$\begin{aligned}
\mathbb{E}\|\Gamma(u)(t) - \Gamma(v)(t)\|_X^2 &\leq c_3 \mathbb{E}\|u(t-r(t)) - v(t-r(t))\|_X^2 \\
&\quad + \frac{1}{1-c_3} N^2 c_f^2 e^{2\beta t}\beta^{-1} t \int_0^t \mathbb{E}\|u(s) - v(s)\|_X^2 ds.
\end{aligned}$$

Hence

$$\sup_{s\in[-\tau,t]} \mathbb{E}\|\Gamma(u)(t) - \Gamma(v)(t)\|_X^2 \leq \alpha(t) \sup_{s\in[-\tau,t]} \mathbb{E}\|u(s) - v(s)\|_X^2$$

where $\alpha(t) = c_3 + \frac{1}{1-c_3} N^2 c_1^2 e^{2\beta t}\beta^{-1} t$.
By condition (iii) in (H4) we have $\alpha(0) = c_3 < 1$. Then there exists $0 < T_1 \leq T$ such that $0 < \alpha(T_1) < 1$ and Γ is a contraction mapping on $S_{T_1}(\varphi)$ and therefore has a unique fixed point, which is a mild solution of Eq. (1.2) on $[-\tau, T_1]$. This procedure can be repeated a finite number of times in order to extend the solution to the entire interval $[-\tau, T]$. This completes the proof.

□

4. Stability

Now, in this section we will analyze the asymptotic behavior of mild solutions to (1.1). We would like to mention that, although it could be carried out a program to study the more general and functional case (1.2), we have preferred to consider this simpler one which, in fact, is the case considered in [5] for the existence of solutions, and will investigate the more general case of (1.2) in a subsequent paper.

Thus, for this purpose we need to assume further assumptions.

(H7) The corresponding resolvent operator $R(t)$ of E.q (2.4) verifies the following : there exists $\lambda > 0,\ M \geq 0$ such that $||R(t)|| \leq Me^{-\lambda t}, \quad \forall t \geq 0.$

(H8) There exist nonnegative real numbers $Q_i \geq 0$ and continuous functions $\xi_i : [0, +\infty) \longrightarrow \mathbb{R}_+$ with $\xi_i(t) \leq P_i e^{-\lambda t}(i = 1, 2), P_i > 0$, such that for all $t \geq 0$ and $x, y \in X$,

$$\|F(t,x)\|_X^2 \leq Q_1\|x\|_X^2 + \xi_1(t), \tag{4.1}$$

$$\|G(t,y)\|_X^2 \leq Q_2\|y\|_X^2 + \xi_2(t). \tag{4.2}$$

(H9) The function $\sigma : [0, +\infty) \longrightarrow \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^{+\infty} e^{\lambda s}\|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds < \infty.$$

Let us recall the following lemma, which is Lemma 3.1 in [9].

Lemma 4.1. *Let $y : [-\tau, +\infty) \longrightarrow [0, +\infty)$ be a function and suppose that there exist some positive constants $\gamma > 0,\ \lambda_1 > 0,\ (i = 1, 2, 3)$ such that :*

$$Y(t) \leq \begin{cases} \lambda_1 e^{-\gamma t} + \lambda_2 \sup\limits_{\theta\in[-r,0]} y(t+\theta) + \lambda_3 \displaystyle\int_0^t e^{-\gamma(t-s)} \sup\limits_{\theta\in[-r,0]} y(s+\theta)ds, & t \geq 0 \\ \lambda_1 e^{-\gamma t}, & t \in [-\tau, 0]. \end{cases}$$

If $\lambda_1 + \dfrac{\lambda_3}{\gamma} < 1$, then $y(t) \leq Me^{-\mu t}, (t \geq -r)$, where μ is a positive root of the algebraic equation $\lambda_2 + \dfrac{\lambda_3}{\gamma - \mu}e^{\mu r} = 1$ and $M = max\left\{\dfrac{\lambda_1(\gamma - \mu)}{\lambda_3 e^{\mu r}}, \lambda_1\right\}$.

The main result of this section is given in the next theorem.

Theorem 4.2. *Suppose that the conditions **(H1)**, **(H2)**, **(H3)'**, **(H4)** - **(H8)** hold and that*

$$k + \frac{4Q_1M^2}{\lambda^2(1-k)} < 1$$

where $k := \sqrt{Q_2}$. Then the mild solution of Eq.(1.2) exponentially decays to zero in mean square. In other words, there exist positive constants $\theta > 0$ and $K(\theta, \varphi) > 0$ such that

$$\mathbb{E}\|u(t)\|_X^2 \leq K(\theta, \varphi)e^{-\theta t} \ ; \quad \forall t \geq -r.$$

Proof. Since

$$k + \frac{4Q_1M^2}{\lambda^2(1-k)} < 1$$

then, it is possible to choose a suitable $\varepsilon > 0$ small enough such that

$$k + \frac{4Q_1M^2}{\lambda(\lambda-\varepsilon)(1-k)} < 1.$$

Let $\delta = \lambda - \varepsilon$ and let $u(t)$ be the mild solution of Eq.(1.2) . For $t \geq 0$, we have

$$\begin{aligned}
\mathbb{E}\|u(t)\|_X^2 &\leq \frac{1}{k}\mathbb{E}\|G(t, u(t-r(t)))\|_X^2 + \frac{4}{1-k}\mathbb{E}\Big\{\|R(t)(\varphi(0) + G(0, \varphi(-r(0))))\|_X^2 \\
&+ \left\|\int_0^t R(t-s)F(s, u(s-\delta(s)))ds\right\|_X^2 \\
&+ \left\|\int_0^t R(t-s)\sigma(s)dB(s)\right\|_X^2\Big\} \\
&\leq \sum_{i=1}^4 I_i(t).
\end{aligned}$$

By condition **(H8)** one easily has

$$\begin{aligned}
I_1(t) &= \frac{1}{k}\mathbb{E}\|G(t, u(t-r(t)))\|_X^2 \\
&\leq \frac{1}{k}\Big\{Q_2\mathbb{E}\|u(t-r(t))\|_X^2 + \xi_2\Big\} \\
&\leq k \quad \mathbb{E}\|u(t-r(t))\|_X^2 + K_1 e^{\delta t}, \qquad (4.3)
\end{aligned}$$

where $K_1 = \frac{P_2}{k}$.

By conditions **(H7)** and **(H8)**, we deduce

$$\begin{aligned}
I_2(t) &\leq \frac{8}{1-k}\mathbb{E}\|R(t)(\varphi(0))\|_X^2 + \frac{8}{1-k}\mathbb{E}\|R(t)(G(0, \varphi(-r(0))))\|_X^2 \\
&\leq \frac{8M^2}{1-k}e^{-2\lambda t}\mathbb{E}\|\varphi(0)\|_X^2 + \frac{8M^2}{1-k}e^{-2\lambda t}\Big\{Q_2\mathbb{E}\|\varphi(-r(0))\|_X^2 + P_2\Big\} \\
&\leq K_2 e^{-\delta t}, \qquad (4.4)
\end{aligned}$$

where $K_2 = \frac{8M^2}{1-K}\left[\mathbb{E}\|\varphi(0)\|_X^2 + \Big\{Q_2\mathbb{E}\|\varphi(-r(0))\|_X^2 + P_2\Big\}\right]$.

Again by conditions **(H7)**, **(H8)** and Hölder's inequality, we obtain

$$\begin{aligned}
I_3(t) &= \frac{4}{1-k}\mathbb{E}\left(\int_0^t Me^{-\lambda(t-s)}\|F(s, u(s-\delta(s)))\|_X ds\right)^2 \\
&\leq \frac{4M^2}{1-k}\int_0^t e^{-\lambda(t-s)}ds \int_0^t e^{-\lambda(t-s)}\Big\{Q_1\mathbb{E}\|u(s-\delta(s))\|_X^2 + \xi_1(s)\Big\}ds \\
&\leq \frac{4Q_1M^2}{\lambda(1-k)}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|u(s-\rho(s))\|_X^2 ds + K_4 e^{-\delta t} \qquad (4.5)
\end{aligned}$$

where $K_4 = \frac{4M^2}{\lambda(1-k)}\frac{P_1}{\lambda - \delta}$.

By virtue of condition (**H6**) and by Lemma 2.2, we derive that

$$\begin{aligned}
I_4(t) &\leq \frac{4M^2}{1-K}2Ht^{2H-1}\int_0^t e^{-2\lambda(t-s)}\|\sigma(s)\|_{\mathcal{L}_2^0}^2\, ds \\
&\leq \frac{4M^2}{1-k}2Ht^{2H-1}\int_0^t e^{-\lambda(t-s)}\|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds \\
&\leq e^{-\delta t}\cdot\frac{4M^2}{1-k}2Ht^{2H-1}e^{-\varepsilon t}\int_0^t e^{\lambda s}\|\sigma(s)\|_{\mathcal{L}_2^0}^2 . ds \qquad (4.6)
\end{aligned}$$

Therefore, condition **(H9)** ensures the existence of a positive constant K_4 such that

$$\frac{4M^2}{1-k}2Ht^{2H-1}e^{-\varepsilon t}\int_0^t e^{\lambda s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds \le K_4 \quad \text{for all} \quad t \ge 0.$$

Then

$$I_4(t) \le K_4\; e^{-\delta t},$$

Inequalities (4.3) - (4.5) and (4.6) together imply that :

$$\mathbb{E}\|u(t)\|^2_X \le \begin{cases} \gamma e^{-\delta t} + k \sup\limits_{-r\le\tau\le 0}\mathbb{E}\|u(t+\tau)\|^2_X + k'\displaystyle\int_0^t e^{-\delta(t-s)}\sup\limits_{-r\le\tau\le 0}\mathbb{E}\|u(s+\tau)\|^2_X ds, & t\ge 0 \\ \gamma e^{-\delta t} & t\in[-\tau,0] \end{cases}$$

where

$$\gamma = \max\left(\sum_{i=1}^4 K_i, \quad \sup_{-r\le\tau\le 0}\mathbb{E}\|\varphi(\tau)\|^2_X\right)$$

and

$$k' = \frac{4Q_1M^2}{\lambda(1-k)}.$$

Since $k + \dfrac{k'}{\delta} < 1$, then it follows from Lemma 4.1 that there exist positive constant $\theta > 0$ and $\tilde{K} > 0$ such that

$$\mathbb{E}\|u(t)\|^2_X \le \tilde{K}e^{-\theta t}\;, \quad \forall t \ge -\tau$$

which is our desired inequality. The proof is thus complete. □

5. Application

Neutral stochastic differential equations arise in many real world problems such as physics, population dynamics, ecology, biological systems, biotechnology, optimal control, theory of elasticity, electrical networks, and so forth. We consider the following stochastic partial neutral functional integro-differential equation with finite delays $r_1, r_2(\infty > r > r_i \ge 0, i = 1,2)$:

$$\begin{cases} \dfrac{\partial}{\partial t}\left[x(t,\xi)+g(t,x(t-r_1,\xi))\right] = \dfrac{\partial^2}{\partial\xi^2}\left[x(t,\xi)+g(t,x(t-r_1,\xi))\right] \\ \qquad + \displaystyle\int_0^t b(t-s)\frac{\partial^2}{\partial\xi^2}\left[x(s,\xi)+g(s,x(s-r_1,\xi))\right]ds \\ \qquad + \tilde{f}(t,x(t-r_2,\xi)) + \sigma(t)\dfrac{dB^H}{dt}(t) \\ x(t,0)+g(t,x(t-r_1,0)) = 0 \quad \text{for} \quad t\ge 0 \\ x(t,\pi)+g(t,x(t-r_1,\pi)) = 0 \quad \text{for} \quad t\ge 0 \\ x(\theta,\xi) = x_0(\theta,\xi), \varphi(s,.)\in L^2[0,T], \quad -r\le\theta\le 0, \quad 0\le\xi\le\pi; \end{cases} \tag{5.1}$$

where B^H denotes a fractional Brownian motion, $g, \tilde{f} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, and $b : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions. Let $Y = L^2([0,\pi])$ and $e_n := \sqrt{\frac{2}{\pi}}\sin(nx)$, $(n = 1,2,3,\cdots)$.

Then $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal basis in Y. Let $X = L^2([0,\pi])$ and $A = \frac{\partial^2}{\partial z^2}$, with domain $D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi])$. Then, it is well known that $Az = -\sum_{n=1}^{\infty} n^2\langle z, e_n\rangle e_n$ for any $z \in X$, and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ on X, which is given by $T(t)\phi = \sum_{n=1}^{\infty} e^{-n^2t}\langle\phi, e_n\rangle e_n$, $\phi \in D(A)$. In order to define the operator $Q : Y \to Y$, we choose a sequence $\{\sigma_n\}_{n\geq 1} \subset \mathbb{R}^+$ and set $Qe_n = \sigma_n e_n$, and assume that $tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty$. Define the process $B^H(s)$ by

$$B^H = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\gamma_n(t)e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\gamma_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent. We suppose that

(1) For $t \geq 0$, $\tilde{f}(t,0) = g(t,0) = 0$.

(2) There exists a positive constant l_1, such that

$$|\tilde{f}(t,\zeta_1) - \tilde{f}(t,\zeta_2)| \leq l_1|\zeta_1 - \zeta_2|$$

for $t \geq 0$ and $\zeta_1, \zeta_2 \in \mathbb{R}$;

(3) There exists a positive constant l_2, such that

$$|\tilde{f}(t,\zeta)| \leq l_2(1 + |\zeta|^2)$$

for $t \geq 0$ and $\zeta \in \mathbb{R}$;

(iv) There exists a positive constant $l_3, 0 < \pi l_3^2 < 1$, such that

$$|g(t,\zeta_1) - g(t,\zeta_2)| \leq l_3|\zeta_1 - \zeta_2|$$

for $t \geq 0$ and $\zeta_1, \zeta_2 \in \mathbb{R}$;

(4) There exists a positive constant l_4, such that

$$|g(t,\zeta)| \leq l_4(1 + |\zeta|^2)$$

for $t \geq 0$ and $\zeta \in \mathbb{R}$;

(5) The function $\sigma : [0, +\infty[\to \mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0}\, ds < \infty, \quad \forall T > 0.$$

(6) There exist nonnegative real numbers $Q_3, Q_4 > 0$ and continuous functions $\xi_3(.),\ \xi_4(.) : [0,+\infty) \longrightarrow \mathbb{R}_+$ with $\xi_i \leq P_i e^{-\lambda t}(i = 3,4), P_i > 0,$ such that $\forall t \geq 0$ and $y \in \mathbb{R}$

$$|\tilde{f}(t,y)|^2 \leq Q_3|y|^2 + \xi_3(t), \tag{5.2}$$

$$|g(t,y)|^2 \leq Q_4|y|^2 + \xi_4(t). \tag{5.3}$$

For $t \geq 0$ and $\phi \in X$, define the operators $F, G : \mathbb{R}^+ \times X \to X$ for $\xi \in [0,\pi]$ by

$$G(t,\phi)(\xi) = g(t,\phi(\xi)), \quad \text{for } \xi \in [0,\pi] \text{ and } \phi \in X,$$

$$F(t,\phi)(\xi) = \tilde{f}(t,\phi(\xi)), \quad \text{for } \xi \in [0,\pi] \text{ and } \phi \in X.$$

If we put

$$\begin{cases} u(t)(\xi) = u(t,\xi) & \text{for } t \geq 0 \text{ and } \xi \in [0,\pi] \\ \varphi(\theta)(\xi) = u_0(\theta,\xi) & \text{for } \theta \in [-r,0] \text{ and } \xi \in [0,\pi], \end{cases}$$

then Eq. (5.1) takes the following abstract form

$$\begin{cases} d\left[u(t)+G(t,u(t-r_1))\right] = A\left[u(t)+G(t,u(t-r_1))\right]dt \\ \qquad + \left[\int_0^t B(t-s)[u(s)+G(s,u(s-r_1))]ds + F(t,u(t-r_2))\right]dt \\ \qquad +\sigma(t)dB^H(t) \ \text{ for } \ t\geq 0, \\ \\ u(t)=\varphi, \quad t\in[-r,0] \end{cases}$$

Moreover, if b is a bounded and C^1 function such that b' is bounded and uniformly continuous, then **(H1)** and **(H2)** are satisfied and hence, by Theorem 2.2, Eq. (5.1) has a resolvent operator $(R(t))_{t\geq 0}$ on X. As a consequence of the continuity of f and g and assumption (1) it follows that F and G are continuous on $\mathbb{R}^+\times X$ with values in X. By assumption (2), one can see that

$$\|F(t,\phi_1)-F(t,\phi_2)\|_X \leq \pi L_1 \|\phi_1-\phi_2\|_X \,,$$

demonstrating that $F(t,u)$ satisfies a Lipschitz condition. Similarly, it can also be verified the same property for $G(t,u)$.
Furthermore, by assumption (iii) it follows that $\|F(t,\phi_1)\|_X \leq l_2\pi(1+\|\phi_1\|_X^2), \qquad t\geq 0.$
The remaining conditions can be verified similarly. Thus, all the assumptions of Theorem 3.3 in [5] and Theorem 4.2 are fulfilled. Therefore, the existence and stability of a unique mild solution of Eq. (5.1) follows.

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