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Functions of time type, curvature and causality theory

C. Atindogbe [a], M. Gutiérrez [b,*], R. Hounnonkpe [a,c]

[a] *Université d'Abomey-Calavi, Benin*
[b] *Universidad de Málaga, Spain*
[c] *Institut de Mathématiques et de Sciences Physiques (IMSP), Benin*

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ABSTRACT

In the present paper various necessary and/or sufficient conditions in terms of the existence and properties of quasi-time functions, semi-time functions and generalized time functions are provided to get different levels of the causal ladder in spacetimes. We also show several links between curvature conditions and causality.

1. Introduction

Causality theory is a well established subject in Lorentzian manifolds and its importance is widely recognized. There are nice results where the levels of the causal ladder can be deduced starting from various assumptions. One of the most classical says that stably causal spacetimes are characterized by the existence of a time function [14]. Recent researches show that different variants of "time-like function" concepts are very useful to deal with causality theory in spacetimes. In [21], chronological and distinguishing spacetime are characterized in terms of volume and generalized time function. Authors in [7,8] established that if a spacetime admits a quasi-time function then it is causal. It is known that any Plane Front Wave spacetime (a generalization of Plane Wave spacetimes) admits a quasi-time function, [3]. Quasi-time functions and other

* Corresponding author.
E-mail addresses: atincyr@gmail.com (C. Atindogbe), m_gutierrez@uma.es (M. Gutiérrez), rhounnonkpe@ymail.com (R. Hounnonkpe).

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notions like semi-time function are also studied and linked to causality by many authors, see [15–19,23,25] and references therein.

Our objective in this work is to contribute to a better understanding of the links between these types of functions and the different levels of the causal ladder of spacetimes. We will use [21] as our main reference. On the other hand, we take advantage of the fact that the techniques used allow us to study the influence of curvature for the same purpose.

In fact, we characterize future and past distinguishing spacetimes with the existence of a lower semi-continuous and upper semi-continuous generalized time function respectively, Theorem 7, improving a result of Minguzzi and Sánchez, [21, Theorem 3.51]. We state in Theorem 11 a relation between the existence of a generalized time function and non total imprisoning spacetimes. We also prove that if there exists a generalized time function on a spacetime and a sequence of semi-time functions which converge pointwisely to it, then the spacetime is strongly causal, Theorem 14. We explore how the topology of the level sets of a semi-time function is related to causality. It is proved that the absence of strong causality implies the spacetime is total imprisoning or admits no semi-time functions all of whose fibers are compact, Theorem 18. Special attention is paid to 3-dimensional spacetimes in Theorems 19 and 20 where sufficient conditions to strong causality and non total imprisoning are described. A special result is Theorem 19 where we combine the existence of a quasi-time function and a curvature condition in the hypothesis. This leads us to study the influence of curvature in causality in the last section. Of course there are many classical examples on the influence of curvature on global properties of the manifold, but it is not usual in causality theory, although there are important examples like classical singularity theorems, see also [9,10]. We impose the null convergence condition and the null completeness condition and explore their effect on non-total imprisoning spacetimes, Theorem 21, on future and past distinguishing spacetimes, on causally simple and globally hyperbolic spacetimes, Corollaries 22, 23 and 24, and on totally vicious spacetimes, Corollary 27 and Theorem 29.

2. Preliminaries

Recall that a spacetime (M, g) satisfies the chronology (causal) condition at a point $p \in M$ provided there are not closed timelike (causal) curves through p. It satisfies the chronology (causal) condition in a subset A if it satisfies the chronology (causal) condition at each point $p \in A$. If $A = M$ we say that (M, g) satisfies the chronology (causal) condition. A spacetime is non-total future imprisoning if no future inextensible causal curve is totally future imprisoned in a compact set. A spacetime is non-partial future imprisoning if no future inextensible causal curve is partially future imprisoned in a compact set. Actually, Beem proved [4, Theorem 4] that a spacetime is non-total future imprisoning if and only if it is non-total past imprisoning, thus in the non-total case one can simply speak of the non-total imprisoning property (condition N, in Beem's terminology [4]). The strong causality condition holds at $p \in M$ provided that given any neighborhood $\mathcal{U}$ of p there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that every causal curve segment with endpoints in $\mathcal{V}$ lies entirely in $\mathcal{U}$. M is strongly causal if the strong causality condition holds at each $p \in M$.

A spacetime (M, g) is future-distinguishing at $p \in M$ if $I^+(p) \neq I^+(q)$ for each $q \in M$, with $q \neq p$. M is future-distinguishing if it is future-distinguishing at every point. This property of being future-distinguishing is called future-distinction. The concept of past-distinction is defined similarly. A spacetime is stably causal if it cannot be made to contain closed trips by arbitrarily small perturbations of the metric. The condition of stable causality is equivalent to the existence of a time function on (M, g), that is to say, a continuous function on M strictly increasing along future directed causal curves. There is one condition, related in some ways to the causality conditions above, which stands, nevertheless, outside the causal ladder.

Definition 1. A spacetime (M, g) is called reflecting if $I^+(q) \subset I^+(p) \Leftrightarrow I^-(p) \subset I^-(q)$ for all $p, q \in M$.

A spacetime (M, g) is called causally continuous if it is reflecting and distinguishing. Causal continuity is stronger than stable causality. A spacetime (M, g) is called causally simple if it is causal and $J^+(p)$, $J^-(p)$ are closed sets for all $p \in M$. Finally, (M, g) is called globally hyperbolic if it is causal and $J^+(p) \cap J^-(p)$ are compact sets for all $p, q \in M$.

3. Functions of time type and causality

Volume and time function are very useful in causality theory, see [6]. They are used to characterize some levels of the causal ladder. For instance, stably causal spacetimes as stated above are characterized by the existence of a time function. Also in [21], chronological and distinguishing spacetime are characterized in terms of volume and generalized time function. In this section we use functions of time type to characterize some causality levels. First, we recall some definitions.

Definition 2. Let (M, g) be a time oriented Lorentzian manifold.

1. A function $f : M \longrightarrow \mathbb{R}$ is a time function if it is continuous and strictly increasing along each future directed nonspacelike curve.
2. A function $f : M \longrightarrow \mathbb{R}$ is a generalized time function if $\forall\, p, q \in M$, $p < q \Rightarrow f(p) < f(q)$.
3. A function $f : M \longrightarrow \mathbb{R}$ is a semi-time function if it is continuous and strictly increasing on future directed timelike curves.
4. A smooth function $f : M \longrightarrow \mathbb{R}$ is said to be a quasi-time function provided
   (a) The gradient of f is past directed nonspacelike, and
   (b) Every null geodesic segment c such that $f \circ c$ is constant, is injective.

Remark 3. A quasi-time function is a semi-time function. Moreover, if a spacetime admits a quasi-time function then it is causal, see [7,8]. It is known that any Plane Front Wave spacetime (a generalization of Plane Wave spacetimes) admits a quasi-time function.

In [19] the author gave the following characterization of distinguishing spacetime.

Theorem 4. *The spacetime (M, g) is future (resp. past) distinguishing if and only if for every $x, z \in M, (x, z) \in J^+$ and $x \in \overline{J^+(z)}$ imply $x = z$ (resp. $(x, z) \in J^+$ and $z \in \overline{J^-(x)}$ imply $x = z$).*

Moreover, in [21, Theorem 3.51] it is proved that a spacetime is past (resp. future) distinguishing if and only if the volume function t^- (resp. t^+) is a generalized time function. Note that t^- is always lower semi-continuous and t^+ always upper semi-continuous. We prove a similar result in the following theorem for generalized time function which are not necessarily volume function. We recall the notion of upper and lower semi-continuity.

Definition 5. A function f on a topological space X is upper (respectively lower) semi-continuous in $x_0 \in X$, if for any $\epsilon > 0$ there exists an open neighborhood U of x_0 such that for any $x \in U$, $f(x) \leq f(x_0) + \epsilon$ (respectively $f(x) \geq f(x_0) - \epsilon$).

Remark 6. If a function f on a topological space X is lower (respectively upper) semi-continuous, then for any sequence $\{x_n\}$ converging to $x \in X$, we have $f(x) \leq \underline{lim} f(x_n)$ (respectively $\overline{lim} f(x_n) \leq f(x)$).

Theorem 7. *A spacetime (M, g) is past (respectively future) distinguishing if and only if it admits a lower (respectively upper) semi-continuous generalized time function.*

Proof. Suppose (M, g) admits an upper semi-continuous generalized time function f and fails to be future distinguishing. Then from Theorem 4, there exist two distinct points $x, z \in M$ such that $(x, z) \in J^+$ and $x \in \overline{J^+(z)}$. Since $(x, z) \in J^+$ we have $f(x) < f(z)$. Also, since $x \in \overline{J^+(z)}$, there exists a sequence $\{x_n\}$ converging to x such that $\forall n$, $x_n \in J^+(z)$. Then we have $\forall n$, $f(z) < f(x_n)$ which implies $f(z) \leq f(x)$ by the upper semi-continuity of f. Contradiction. So if (M, g) admits an upper semi-continuous generalized time function then it is future distinguishing. For the converse, if (M, g) is future distinguishing then from [21, Lemma 3.39 and Theorem 3.51] it admits an upper semi-continuous generalized time function.

The past distinguishing case is shown similarly. □

Corollary 8. *A spacetime (M, g) is distinguishing if and only if it admits a lower semi-continuous generalized time function and an upper semi-continuous generalized time function.*

We need to recall some basic definitions from dynamical system.

Definition 9. Let M be a manifold, X a complete vector field on M and Φ its flow. Let $\gamma : \mathbb{R} \longrightarrow M$ be an integral curve of X. The sets

$$\omega(\gamma) = \{p \in M : \gamma(t_n) \to p; t_n \to \infty\}$$

and

$$\alpha(\gamma) = \{p \in M : \gamma(t_n) \to p; t_n \to -\infty\}$$

are called respectively the ω-limit set and the α-limit set of the orbit γ. A point p is called positively recurrent if $p \in \omega(\gamma_p)$ and it is called negatively recurrent if $p \in \alpha(\gamma_p)$, where γ_p is the unique integral curve of X through p. A subset $A \subset M$ is invariant if $\Phi_t(A) \subset A, \forall t \in \mathbb{R}$. It is known that for any integral curve γ, $\omega(\gamma)$ and $\alpha(\gamma)$ are closed (probably empty) invariant subsets. A closed, non-empty, invariant subset $A \subset M$ is a minimal set if it contains no proper, closed, non-empty, invariant subset.

Our arguments will make intensively use of the following Lorentzian null splitting theorem due to Galloway.

Theorem 10. *[11, Theorem IV.1] Let M be a null geodesically complete spacetime which obeys the null convergence condition and contains a null line η. Then η is contained in a smooth achronal totally geodesic null hypersurface S.*

It is obvious that if a spacetime admits a generalized time function then it is causal. However the relation between the existence of generalized time function and non totally imprisonment is not clear.

Theorem 11. *Let (M, g) be a spacetime admitting a generalized time function f and a complete timelike conformal vector field. Then (M, g) is non totally imprisoning.*

Proof. Suppose (M, g) is totally imprisoned. Since (M, g) is chronological, it contains a null line η contained in a compact minimal invariant set Ω such that $\bar{\eta} = \Omega$ and through each point of Ω, there passes one and only one null line contained in Ω, [16, Theorem 3.9]. Moreover, take any point $p \in \eta$, then there exists $t_n \to \infty$ such that $\eta(0) = p$ and $\eta(t_n) \longrightarrow p$. Let ζ be a conformal timelike complete vector field on M with flow ϕ_t. Then, $\forall t \in \mathbb{R}$, $\phi_t \circ \eta$ is a causal curve. Let γ_p denote the integral curve of ζ such that $\gamma_p(0) = p$. As f is a generalized time function,

$$f \circ \gamma_p : \mathbb{R} \longrightarrow \mathbb{R}$$

is strictly increasing and so it is easy to see that it is continuous outside a countable set. Let $t_0 \in \mathbb{R}$ be such that $f \circ \gamma_p$ is continuous at t_0. From [23, Proposition A.1], f is continuous at $\gamma_p(t_0) = q$. The causal curve $\phi_{t_0} \circ \eta$ satisfies $(\phi_{t_0} \circ \eta)(0) = q$ and $(\phi_{t_0} \circ \eta)(t_n) \longrightarrow q$. It follows that $f((\phi_{t_0} \circ \eta)(t_n))$ is strictly increasing and converge to $f(q)$. The contradiction follows from the fact that $f((\phi_{t_0} \circ \eta)(0)) = f(q)$. $\square$

Remark 12. The following example shows that the hypothesis on the existence of a generalized time function can not be weakened to the property of the spacetime to be causal. Let $\mathbb{T}^2$ be the two-torus given as the coordinate patch $(x, y) \in [0, 1] \times [0, 1] \subset \mathbb{R}^2$ with the identifications $(0, y) \sim (1, y)$ for all $y \in [0, 1]$ and $(x, 0) \sim (x + \sqrt{2}, 1)$. The orbits of the vector field ∂_y are dense in $\mathbb{T}^2$ as $\sqrt{2}$ is irrational. We now consider the spacetime $(\mathbb{R} \times \mathbb{T}^2, g)$ with the Lorentzian metric

$$g = -dt^2 + 2dtdy + dx^2.$$

Then, $(\mathbb{R} \times \mathbb{T}^2, g)$ is stationary with Killing vector field ∂_t, and every $t =$ constant slice $\{t\} \times \mathbb{T}^2$ contains imprisoned but non-closed causal curves, i.e., the spacetime is causal but totally imprisoning.

We need the following characterization of strongly causal spacetimes, [19].

Theorem 13. *The spacetime (M, g) is strongly causal if and only if for every $x, z \in M, (x, z) \in J^+$ and $(z, x) \in \overline{J^+}$ imply $x = z$.*

This allows us to give a link between strong causality and time-like functions.

Theorem 14. *Let (M, g) be a spacetime. Suppose there exists a generalized time function f and a sequence $\{f_k\}$ of semi-time functions which converge pointwisely to f. Then (M, g) is strongly causal.*

Proof. Suppose (M, g) is not strongly causal. Then by Theorem 13, there exists two distinct points $x, z \in M$ such that $(x, z) \in J^+$ and $(z, x) \in \overline{J^+}$. Since $(x, z) \in J^+$ we have $f(x) < f(z)$. Also, since $(z, x) \in \overline{J^+}$, there exists two sequences $\{x_n\}$ and $\{z_n\}$ converging respectively to x and z such that $\forall n$, $x_n \in J^+(z_n)$. Then we have $\forall k$ and $\forall n$, $f_k(z_n) \leq f_k(x_n)$ which implies by the continuity of each f_k that $f_k(z) \leq f_k(x)$ $\forall k$. Take the limit as k goes to infinity and get $f(z) \leq f(x)$. Contradiction. $\square$

Remark 15.

1. As shown in the proof, the functions f_k can be taken just non decreasing on future directed causal curve. Moreover, if the functions f_k are not continuous but lower semi-continuous (respectively upper semi-continuous) and non decreasing on future directed causal curve then by similar arguments and using Theorem 4, it can be shown that (M, g) is past distinguishing (respectively future distinguishing).
2. Suppose (M, g) is a stably causal spacetime. Then it admits a (continuous) time function f. If we take $f_k = f$ $\forall k$ in Theorem 14, we rediscover the well known fact that any stably causal spacetime is strongly causal.

We explore now how the topology of the fibers of a semi-time function is related to causality. The following theorem is needed.

Theorem 16. *[20] Let $c : [a, b] \to M$ be a maximizing causal curve on the spacetime (M, g), then there are the following possibilities,*

1. *c is timelike and (M, g) is strongly causal at $c(t)$ for every $t \in (a, b)$.*

2. *c is lightlike and one of the following possibilities holds*
   (a) *(M, g) is strongly causal at $c(t)$ for every $t \in (a, b)$.*
   (b) *Strong causality is violated at every point of c, and $(c(t_2), c(t_1)) \in \overline{J^+}$ for all $t_1, t_2 \in [a, b]$ with $t_1 < t_2$.*
   (c) *c intersects the closure of the chronology violating set at some point $x = c(t), t \in (a, b)$. Moreover, all the points in $c((a, b))$ at which strong causality is violated belong to the closure of the chronology violating set. In particular, (M, g) is not chronological.*

Remark 17. This theorem is an extension of a result of Newman, [22, Proposition 3.5], which states that given a maximizing timelike segment $c : [a, b] \to M$, the spacetime (M, g) is strongly causal at (a, b). The case (ii) in short states that outside the closure of the chronology violating set, the maximizing lightlike segments propagate the property of strong causality.

Theorem 18. *A non total imprisoning spacetime admitting a semi-time function with compact level sets is strongly causal.*

Proof. Assume strong causality fails at some $x \in M$. Then there exists a null line $\gamma : I \to M$ passing through x, [18]. Without loss of generality, we can suppose γ future directed. Call f the semi-time function. We show that f is constant on γ. It holds $\gamma(t_1) < \gamma(t_2)$ for all $t_1, t_2 \in I$ with $t_1 < t_2$. So we have $f(\gamma(t_1)) \leq f(\gamma(t_2))$. Moreover, it holds $(\gamma(t_2), \gamma(t_1)) \in \overline{J^+}$ from the item 2.b of the above theorem. Then, there exists two sequences $\{x_n\}$ and $\{z_n\}$ converging respectively to $\gamma(t_1)$ and $\gamma(t_2)$ such that $\forall n$, $x_n \in J^+(z_n)$ and consequently $f(z_n) \leq f(x_n)$ for all n. By the continuity of f it holds $f(\gamma(t_2)) \leq f(\gamma(t_1))$. This proves that for all $t_1, t_2 \in I$, $f(\gamma(t_2)) = f(\gamma(t_1))$ and then $f \circ \gamma$ is constant, that is, γ is contained in some level set of f. Since f has compact level sets, γ is a totally imprisoned causal line, which contradicts the non total imprisonment condition. $\square$

Now we focus on 3-dimensional spacetimes, where the existence of a semi-time function can be slightly weakened.

Theorem 19. *Let (M, g) be a null complete 3-dimensional spacetime manifold satisfying the null convergence condition. Suppose it admits a quasi-time function f with compact connected level sets non diffeomorphic to a torus. Then (M, g) is strongly causal.*

Proof. We prove first that (M, g) is non total imprisoning. Suppose (M, g) is totally imprisoning. The existence of a quasi-time function implies that (M, g) is causal (Remark 3) in particular chronological. Hence from [16, Theorem 3.9], it contains a null line η contained in a compact minimal invariant set Ω such that $\bar{\eta} = \Omega$ and through each point of Ω, there passes one and only one null line contained in Ω. Using the null completeness and the null convergence condition, η is contained in a smooth (topologically) closed achronal totally geodesic null surface L, Theorem 10, which can be taken connected. Take a timelike vector field as a rigging for L and rescale its associated rigged vector field ξ such that ξ is complete, see [12]. Then, as any null line through a point $p \in \Omega$ shares the same trace as the integral curve of ξ through p, it follows that Ω is a compact minimal invariant set of the flow of ξ. It is known that a compact minimal set of a C^2 differentiable dynamical system on a 2-surface S is either a fixed point, a periodic orbit or all of S, in which case S is a torus, see [24], (although in the original paper [24] S must be compact, it is not necessary, see [13, Ch. VII, 12.1.]). Since ξ is a lightlike vector field and (M, g) is causal, the compact minimal set Ω is neither a fixed point nor a periodic orbit. So $\Omega = L$ and then L is a compact null surface which is a torus. We have $\bar{\eta} = \Omega$ which implies that strong causality fails on η. Following the same arguments as in the proof of Theorem 18, η is contained in a level set say F of f which is connected. Since $\bar{\eta} = \Omega = L$ and

F is closed then L is contained in F. Note that F is a smooth surface. It follows that L is an open subset of F. But being compact L is also closed in F. From the connectedness of F we get that $F = L$ which gives the contradiction as the level sets of f are not torus. This proves that (M, g) is non total imprisoning. Finally since f is a quasi-time function and hence a semi-time function with compact fibers, Theorem 18 states that (M, g) is strongly causal and the proof is complete. □

If a quasi-time function f exists on a spacetime (M, g) then it is causal. A natural question is under which condition (M, g) can be non total imprisoning. The following result gives an answer.

Theorem 20. *Let (M, g) be a null complete 3-dimensional spacetime manifold satisfying the null convergence condition. Suppose it admits a quasi-time function f with lightlike gradient and non compact connected fibers. Then (M, g) is non total imprisoning.*

Proof. We argue as in Theorem 19. Suppose (M, g) is totally imprisoned. The existence of a quasi-time function implies that (M, g) is causal (Remark 3) in particular chronological. Hence from [16, Theorem 3.9], it contains a null line η contained in a compact minimal invariant set Ω such that $\bar{\eta} = \Omega$ and through each point of Ω, there passes one and only one null line contained in Ω. Using the null completeness and the null convergence condition, η is contained in a smooth (topologically) closed achronal totally geodesic null surface L, Theorem 10, which can be taken connected. As strong causality fails on η, following the same arguments as in the proof of Theorem 18, η is contained in a level set say F of f. Note that F is a smooth lightlike null surface since the gradient ∇f is lightlike. Moreover, ∇f is tangent to F. Take a timelike vector field as a rigging for F and rescale its associated rigged vector field ξ such that ξ is complete, [12], then Ω is a compact minimal invariant set of the flow of ξ. Since ξ is a lightlike vector field and (M, g) is causal, the compact minimal set Ω is neither a fixed point nor a periodic orbit. So $\Omega = L$ and then L is a compact null surface which is a torus. Since $\bar{\eta} = \Omega = L$ and F is closed then L is contained in F. It follows that L is an open subset of F. But being L compact, it is also closed in F. From the connectedness of F we get that $F = L$ which gives the contradiction as the level sets of f are non compact. This proves that (M, g) is non total imprisoning. □

4. Curvature conditions

Theorem 19 above shows a link between curvature and causality which as we said, is not usual in the literature. This justifies to explore further curvature and causality interrelations, and this is the aim of this section.

Null convergence condition and null completeness have been used intensively in the formulation of numerous results in general relativity, in particular singularity theorems. We prove that if both of these conditions hold in a spacetime then the condition for the spacetime to be causal in the definition of causally simple and globally hyperbolic spacetime can be weakened to chronological. Moreover we prove that if both condition hold and $J^+(p)$ (resp. $J^-(p)$) is closed for any $p \in M$, then the spacetime is either future distinguishing (resp. past distinguishing) or it is non chronological and in this case all the connected components of the boundary of the chronological violating set are non compact. In particular, if null convergence condition and null completeness hold on a compact spacetime which satisfies $J^+(p)$ closed or $J^-(p)$ closed for any $p \in M$, then it is totally vicious. Others related results are also proved. We begin with a result concerning non total imprisonment.

Theorem 21. *Let (M^n, g), with $n \geq 3$ be a chronological, null geodesically complete spacetime which obeys the null convergence condition. Suppose for all $p \in M$, $J^+(p)$ is closed or $J^-(p)$ is closed. Then the spacetime is non total imprisoning.*

Proof. Suppose (M, g) is totally imprisoning. Since (M, g) is chronological it follows from [16, Theorem 3.9] that it contains a null line η contained in a compact minimal invariant set Ω such that $\bar{\eta} = \Omega$. Moreover, all the points belonging to Ω share the same chronological past and future. Using the null completeness and the null convergence condition, η is contained in a smooth (topologically) closed, achronal and totally geodesic null hypersurface S, Theorem 10. More precisely, S is the connected component of $\partial I^+(\eta)$ (or of $\partial I^-(\eta)$) containing η. Since all points belonging to Ω share the same chronological past and future, given any $p \in \eta$ we have $\partial I^+(\eta) = \partial I^+(p) = \partial J^+(p)$ and $\partial I^-(\eta) = \partial I^-(p) = \partial J^-(p)$. By hypothesis, $J^+(p)$ is closed or $J^-(p)$ is closed. Assume the former case. Using the fact that $J^+(p)$ is closed, it holds $\partial J^+(p) = E^+(p)$. Recall that any point of $E^+(p)$ lies on a null geodesic segment from p and $E^+(p)$ is always connected. So we get that $S = E^+(p)$. The contradiction follows from the fact that $E^+(p)$ cannot be a smooth null hypersurface because it contains the initial portion of the null cone at p. Now, if $J^-(p)$ is closed then by a similar argument we have that $S = E^-(p)$ and the contradiction follows as above. It follows that (M, g) is non total imprisoning. □

Corollary 22. *Let (M^n, g), with $n \geq 3$ be a chronological, null geodesically complete spacetime which obeys the null convergence condition. If $J^+(p)$ (resp. $J^-(p)$) is closed for all $p \in M$, then the spacetime is future distinguishing (resp. past distinguishing).*

Proof. We give the proof in the case that $J^+(p)$ is closed for all $p \in M$. The past case is similar. From Theorem 21, (M, g) is non total imprisoning and in particular it is causal.

Suppose (M, g) failed to be future distinguishing. Then there exist distinct points $p, q \in M$ such that $I^+(p) = I^+(q)$ and from the closedness of J^+ we have $p \in \overline{I^+(p)} = \overline{I^+(q)} = \overline{J^+(q)} = J^+(q)$ and similarly $q \in J^+(p)$. This means that there is a closed causal curve through p. Contradiction. □

We recall that a spacetime (M, g) is causally simple if it is causal and $J^+(p), J^-(p)$ are closed sets for all $p \in M$ and globally hyperbolic if it is causal and $J^+(p) \cap J^-(p)$ are compact sets for all $p, q \in M$, [5]. It is known that in the above two definitions, the condition for the spacetime to be causal can not be in general relaxed to the chronological condition, see [21, Remark 3.72]. The next two Corollaries show that under the association of the null convergence condition and null completeness the causal condition can be relaxed to the chronological one.

Corollary 23. *Let (M^n, g), with $n \geq 3$ be a chronological, null geodesically complete spacetime which obeys the null convergence condition. If $\forall p \in M$, $J^+(p)$ and $J^-(p)$ are closed then the spacetime is causally simple.*

Proof. By hypothesis $J^+(p)$ and $J^-(p)$ are closed sets for all $p \in M$. From Theorem 21, (M, g) is causal. So (M, g) is causally simple. □

We have also the following.

Corollary 24. *Let (M^n, g), with $n \geq 3$ be a chronological, null geodesically complete spacetime which obeys the null convergence condition. If $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$. Then the spacetime is globally hyperbolic.*

Proof. Since $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$ it follows that $J^+(p)$ and $J^-(p)$ are closed sets for all $p \in M$, see [21, Proposition 3.71]. From Theorem 21, (M, g) is causal and then it is globally hyperbolic. □

Now, we consider the case when the spacetime is non chronological. The chronology violating set is $\mathcal{C} = \{x : x \ll x\}$, and is made by all the events through which there passes a closed timelike curve. The spacetime violates chronology if $\mathcal{C} \neq \emptyset$, that is, if there is a closed timelike curve. Suppose $\mathcal{C} \neq \emptyset$, then

$\mathcal{C}$ splits into equivalence classes according to Carter's equivalence relation $x \sim y \Leftrightarrow x \ll y$ and $y \ll x$. Two points belong to the same class if there is a closed timelike curve passing through them. The class of $x \in \mathcal{C}$ is denoted $[x]$. Note that $[x] = I^+(x) \cap I^-(x)$, thus $[x]$ is open. So the chronological violating set can be written $\mathcal{C} = \bigcup_\alpha \mathcal{C}_\alpha$, with $\mathcal{C}_\alpha$ its (open) connected components. The boundary of the component $\mathcal{C}_\alpha$ can be written $\partial\mathcal{C}_k = \bigcup_\alpha B_{\alpha k}$, with $B_{\alpha k}$ its (closed) connected components. Some authors have studied the compactness of the components of the chronological violating set's boundary with respect to some energy condition ([15]) or absence of null line ([18]). More precisely, we have the following Kriele's theorem.

Theorem 25. *Suppose that (M, g) satisfies the null energy condition and the null genericity condition. If a connected component of the boundary of the chronology violating set $\mathcal{C}$ is compact, then (M, g) is null geodesically incomplete.*

The following theorem shows that the conclusion in Kriele's theorem holds if the null genericity condition is replaced by the condition: for all $p \in M$, $J^+(p)$ is closed or $J^-(p)$ is closed.

Theorem 26. *Let (M^n, g), with $n \geq 3$ be a spacetime which obeys the null convergence condition. Suppose $J^+(p)$ or $J^-(p)$ is closed for all $p \in M$. If a connected component of the boundary of the chronological violating set is non empty and compact, then (M, g) is null geodesically incomplete.*

Proof. Suppose (M, g) is null geodesically complete. Since a connected component (say B) of the boundary of the chronological violating set is compact, there exists a null line η contained in the compact set B. Note that B do not meet the chronological violating set since the latter is open. So from [16, Theorem 3.9], (M, g) contains a compact minimal invariant set Ω such that $\bar{\eta} = \Omega$. Moreover, all the points belonging to Ω share the same chronological past and future. Using the null completeness and the null convergence condition, η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface S, Theorem 10. More precisely, S is the connected component of $\partial I^+(\eta)$ (or of $\partial I^-(\eta)$) containing η. Using the fact that for all $p \in M$, $J^+(p)$ is closed or $J^-(p)$ is closed, we get the contradiction as in the proof of Theorem 21. □

Corollary 27. *Let (M^n, g), with $n \geq 3$ be a compact spacetime which obeys the null convergence condition and is null complete. Suppose for all $p \in M$, $J^+(p)$ is closed or $J^-(p)$ is closed, then (M, g) is totally vicious. In particular, any compact flat spacetime which satisfies $J^+(p)$ closed for all $p \in M$ (respectively $J^-(p)$ closed for all $p \in M$) is totally vicious.*

Proof. Suppose (M, g) is non totally vicious. Then since it is a compact spacetime, each connected component of the boundary of the chronological violating set would be non empty and compact. This contradicts the null completeness assumption, see Theorem 26. For the last assertion, note that any compact flat spacetime is complete and satisfies the null convergence condition. □

We finish with a theorem for which we need a technical proposition which uses the rigging technique, see [12] for details and [1,2] for further developments.

Proposition 28. *Let (M^{2n+1}, g) be a time orientable Lorentzian manifold admitting a gradient spacelike vector field with only two critical points. Then (M, g) contains no compact null hypersurface.*

Proof. Suppose a compact null hypersurface L in (M, g). We can take any timelike vector field ζ on M as a rigging for L. Let ∇f be a spacelike gradient vector field on M with two critical points. Let h denote the restriction of f on L. Then, along M it holds $\nabla f_{|L} = X + a\xi + bN$, where a and b are smooth functions on M and $X \in \mathcal{S}(\zeta)$, the screen distribution induced by ζ. Using the associated rigged metric $\widetilde{g}$, we get $\widetilde{\nabla} h = X + b\xi$. Let p be a critical point of h on M. Then $\widetilde{\nabla} h_{|_p} = 0$ and $\nabla f_{|_p} = a(p)\xi_p$. Since ∇f is spacelike we get p is a critical point of ∇f. As f has only two critical points, it follows that h has just two critical points since L is compact. Let us recall that a compact manifold admitting a function with only two critical points is homeomorphic to the euclidean sphere, so being L a hypersurface, it is homeomorphic to $\mathbb{S}^{2n}$ in contradiction with the fact that L has zero Euler characteristic since the rigged vector field is a global non vanishing vector field. □

Theorem 29. *Let (M^n, g), with $n \geq 3$ be a compact spacetime which obeys the null convergence condition and is null complete. Suppose there exists a gradient spacelike vector field with only two critical points. Then n is odd, M is homeomorphic to the euclidean sphere $\mathbb{S}^n$ and (M, g) is totally vicious.*

Proof. We will prove that (M, g) contains no null line and then we get the conclusion that (M, g) is totally vicious, see [18, Theorem 12]. Suppose (M, g) contains a null line. Then using the null completeness and the null convergence condition, this null line is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface L, Theorem 10. As M is compact and L is (topologically) closed, it follows that L is a compact null hypersurface. Moreover since (M, g) carries a function with only two critical points, it is homeomorphic to $\mathbb{S}^n$. It is well known that compact manifold with non zero Euler characteristic can not carry a Lorentzian metric. So M is homeomorphic to an odd dimensional sphere. But from Proposition 28, there can not exist compact null hypersurface, which gives the contradiction. □

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References

[1] C. Atindogbe, M. Gutiérrez, R. Hounnonkpe, New properties on normalized null hypersurfaces, Mediterr. J. Math. 15 (2018) 166.
[2] C. Atindogbe, M. Gutiérrez, R. Hounnonkpe, Correction to: New properties on normalized null hypersurfaces, Mediterr. J. Math. 15 (2018) 209.
[3] J.K. Beem, P.E. Ehrlich, Global Lorentzian Geometry, Marcel Dekker Inc., New York, 1981.
[4] J.K. Beem, Conformal changes and geodesic completeness, Comm. Math. Phys. 49 (1976) 179–186.
[5] A.N. Bernal, M. Sánchez, Globally hyperbolic spacetimes can be defined as "causal" instead of "strongly causal", Class. Quantum Grav. 24 (2007) 745–749.
[6] A.N. Bernal, M. Sánchez, Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes, Comm. Math. Phys. 257 (2005) 43–50.
[7] P.E. Ehrlich, G. Emch, Quasi-Time Functions in Lorentzian Geometry, Lecture Notes in Pure and Appl. Math., vol. 144, Marcel Dekker Inc., 1993, pp. 203–212.
[8] P.E. Ehrlich, G. Emch, Gravitational waves and causality, Rev. Math. Phys. 4 (1992) 163–221, Rev. Math. Phys. 4 (1992) 501 (Errata).
[9] J.L. Flores, M. Sánchez, Geodesic connectedness and conjugate points in GRW space-times, J. Geom. Phys. 36 (2000) 285–314.
[10] J.L. Flores, M. Sánchez, Causality and conjugate points in general plane waves, Classical Quantum Gravity 20 (2003) 2275–2291.
[11] G.J. Galloway, Maximum principles for null hypersurfaces and null splitting theorems, Ann. Henri Poincaré 1 (2000) 543–567.
[12] M. Gutiérrez, B. Olea, Induced Riemannian structures on null hypersurfaces, Math. Nachr. 289 (2016) 1219–1236.
[13] P. Hartman, Ordinary Differential Equations, John Wiley & Sons, New York-London-Sydney, 1964.
[14] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-Time, Cambridge University Press, 1973.
[15] M. Kriele, The structure of chronology violating sets with compact closure, Classical Quantum Gravity 6 (1989) 1607–1611.
[16] E. Minguzzi, Non-imprisonment conditions on spacetime, J. Math. Phys. 49 (2008) 062503, 9 pp.
[17] E. Minguzzi, Weak distinction and the optimal definition of causal continuity, Classical Quantum Gravity 25 (2008) 075015, 7 pp.
[18] E. Minguzzi, Chronological spacetimes without lightlike lines are stably causal, Comm. Math. Phys. 288 (2009) 801–819.
[19] E. Minguzzi, The causal ladder and the strength of K-causality. I, Classical Quantum Gravity 25 (2008) 015009, 13 pp.
[20] E. Minguzzi, Limit curve theorems in Lorentzian geometry, J. Math. Phys. 49 (2008) 092501, 18 pp.

[21] E. Minguzzi, M. Sánchez, The causal hierarchy of spacetimes, in: D.V. Alekseevsky, H. Baum (Eds.), Recent Developments in Pseudo-Riemannian Geometry, in: ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, pp. 299–358.

[22] R.P.A.C. Newman, A proof of the splitting conjecture of S.-T. Yau, J. Differ. Geom. 31 (1990) 163–184.

[23] A. Rennie, B.E. Whale, Generalised time functions and finiteness of the Lorentzian distance, J. Geom. Phys. 106 (2016) 108–121.

[24] A.J. Schwartz, A generalization of a Poincaré-Bendixson theorem to closed two-dimensional manifolds, Am. J. Math. 85 (1963) 453–458.

[25] H.J. Seifert, Smoothing and extending cosmic time functions, Gen. Relativ. Gravit. 8 (1977) 815–831.