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Extremal Copulas and Tail Dependence in Modeling Stochastic Financial Risk

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Abstract. These last years the stochastic modeling became essential in financial risk management related to the ownership and valuation of financial products such as assets, options and bonds. This paper presents a contribution to the modeling of stochastic risks in finance by using both extensions of tail dependence coefficients and extremal dependance structures based on copulas. In particular, we show that when the stochastic behavior of a set of risks can be modeled by a multivariate extremal process a corresponding form of the underlying copula describing their dependence is determined. Moreover a new tail dependence measure is proposed and properties of this measure are established.

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Key Words and Phrases: Copulas, stochastic finance, extreme values theory, stochastic process, tail dependence, Value at risk.

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1. Introduction

Now a days, stochastic modeling in finance has become essential in risks management related to the ownership and valuation of financial products (assets, options, bonds,etc.). Several models exist in financial literature including the Markowitz [17] model, that of Black-Scholes-Merton [5] and the model of Heston [15], see [17]. In the Markowitz model, the variance (assumed to be constant) is used as a risk measure to determine the optimal portfolio. The BSM model based on the use of stochastic differential equations, provides a formula for valuing options and bonds. This model includes several assumptions such as constant volatility and a constant and deterministic interest rate. The model of Heston based on diffusion processes, gives a semi-analytical formula for some derivative products with a certain realism . This model is part of stochastic volatility models which differs from BSM. Despite their wide use, all of these models are built under the Gaussian hypothesis

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which does not capture the leptokurticity effect in stochastic finance and there are also weaknesses when we want to model the joint distribution. To overcome these problems, other models developped on the basis of on extreme values theory (EVT) and copulas theory, are proposed. The EVT provides a framework for better modeling leptokurticity and copulas gives better link between marginal and joint distribution, if one is confronted with the study of several risks. Indeed, the copula makes it possible to capture the structure of dependence which exists between several random variables. The first uses of copulas in financial modeling are very recent (see [4]). They are more and more studied because of their flexibility and their ease of interpretation and of implementation. For more details, see Embrechts[11]; Malavergne[16] and the references therein.

The essentiels results in EVT are due to Fisher-Tippet (theorem of three types) [13] and Balkema-de Haan- Pickands [1]. These results establish in particular that, if $X_1; \dots; X_n$ is a sequence of random variables with common distribution F, then the excess variable $\{Y_j = X_j - u/X_j > u\}$; $N_u = card\{k/X_k > u\}$ which is governed by the law F_u (conditional distribution of the unknown continuous distribution function F with respect to the threshold u), converges asymptotically towards a non-degenerate law. According to Pickands, Balkema and de Haan: when the threshold u tends towards the right endpoint x_F, it follows that :

$$\lim_{u \to x_F} |F_u(y) - H(y)| = 0.$$

where the function $H(y)$ corresponds to the distribution function of the generalized Pareto law(GPD) :

$$H_{\xi,\sigma}(y) = \begin{cases} 1 - [1 + \xi \frac{y}{\sigma}]^{-\frac{1}{\xi}}, & \xi \neq 0 \\ \\ 1 - \exp(-\frac{y}{\sigma}), & \xi = 0 \end{cases} ; \tag{1}$$

where $\{\xi \in \mathbb{R}\}$ is the tail index and $\{\mu \in \mathbb{R}\}$, $\{\sigma \succ 0\}$ are localization and dispersion parameters respectively; and $y \in [0; x_F - u]$ $, if\ \xi \geq 0$ and $y \in [0, -\frac{\sigma}{\xi}]$ $, if\ \xi < 0$.[†]

Risk is omnipresent in any kind of investment operation on the financial market. Its universe keeps growing due to the growing creation of new financial products and the emergence of financial markets. In this context, management practicers have been strengthened especially in matters of regulation (Basel I, II then III), recovery (see [10]) and in matters design of risk measurement tools. These risk measurement tools are diverse (depending on the nature of the risk) and various. The main theoretical risk measures are volatility, correlation coefficient, beta coefficient, VaR, TVaR, CVaR, Kendall tau, Spearman rhô, the tail dependence coefficient. For more details on the typology of risk and the application of these measures in finance, see [11],[16].

Since its first use by JP Morgan (1990), the VaR is currently the most used in finance (in the univariate case), due to its simplicity of interpretation and calculation. Indeed,

[†]the GPD H can be written in the form: $H(y) = 1 + \log G(y)$ where G corresponds to the Generalized Extreme Value (GEV) distribution. Considering the GEV model with localization parameter $\mu = 0$ (for the excesses the effect of the localization parameter is taken in account in the sequence $(a_n)_{n \geq 1}$).

if X is a variable modeling the gain (or loss) with distribution F then the VaR at the threshold α is defined by:

$$VaR_{\alpha}(X) = \inf\left[x/P(X \geq x) \leq \alpha\right] = F^{-1}(\alpha). \tag{2}$$

Several versions of this measure are proposed in the multivariate framework in Embrechts [11] and Garcin et al.[14]. To better assess the risk dynamics of a portfolio (the risk linked to holding the asset portfolio, under various conditions and over time), it is often convenient to model the dynamics of assets (and therefore of the portfolio) by a stochastic process. The stochastic process thus defined will be assimilated to the stochastic risk which we want to study and quantify the magnitude of the danger.

The objective of this article is to investigate, first, the dependence structure of multivariate extremal processes. Then, we introduced a new measure of multivariate tails dependences (lower and upper). Many interesting properties are established. The rest of the paper is organized as follows: in section 2, the essentiels concepts to the study are recalled and in section 3 the main results obtained are presented and section 4 we give a conclusion and discussion.

2. Preliminaries

In this section, we collect essentials notions, definitions and properties on copulas, tail dependence coefficients and extremal processes, which will be necessary for our approach. For more details, the reader are refered to authors such as Schimitz [20] which offers a well-developed framework for the analysis of stochastic processes by copulas or Nelsen [18] who gave an introduction to copulas and their statistical and mathematical foundations.

2.1. Survey of Copulas of Multivariate Processes

Copulas provide a natural way to construct multivariate distribtutions whose marginals are uniform and not necessarily exchangeable.

Definition 1. *[19] Let $X = (X_1, ..., X_n)$ be a random vector with multivariate continuous distribution function (c.d.f.) F and c.d.f marginal $F_1, ..., F_n$. The copula of X (or the c.d.f. F respectively) is the multivariate c.d.f. C of the random vector $U = [F_1(X_1), ..., F_n(X_n)]$. Due to the continuity of $\{F_i, 1 \leq i \leq n\}$, each component of U is standard uniformly distributed, i.e., $U_i \backsim U(0,1)$ for $i = 1, ..., n$.*

Particularly, every n-copula must satisfy the n-increasing property. That means that, for any rectangle $B = [a,b]^n \subseteq \mathbb{R}^n$, the B-volume C_B of C is positive, that is,

$$C_B = \int_B dC\left(u\right) = \sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} \left(-1\right)^{i_1+...+i_n} C\left(u_{1,i_1}; ...; u_{1,i_n}\right) \geq 0. \tag{3}$$

for all $(u_{1,1};...;u_{n,1})$ and $(u_{1,2};...;u_{n,2}) \in [0,1]^n$ with $u_{i,1} \leq u_{i,2}$, for $i = 1, ..., n$.

Moreover, from Definition 1, it yields the following parameterization of F (see [4] and [20]), for $(x_1, ..., x_n) \in \bar{\mathbb{R}}^n$ (where $\bar{\mathbb{R}} = [-\infty, +\infty]$)

$$F(x_1, ..., x_n) = C\left[F_1(x_1), ..., F_n(x_n)\right] . \tag{4}$$

This result makes possible to join the marginal to the multivariate joint distribution. Especially in the survival analysis (finance or biostatistics) the above relation, gives the survival copula in function of the survival law $\bar{F}$ of F by:

$$\bar{C}(\bar{F}_1(x_1), ..., \bar{F}_n(x_n)) = \bar{F}(x_1, ..., x_n). \tag{5}$$

The survival copula $\bar{C}$ is linked to the copula C, for all $(u_1, ..., u_n) \in [0,1]^n$, by:

$$\bar{C}(u_1, u_2, ..., u_n) = \sum_{M \subset N} (-1)^m C\left[(1-u_1)^{1_1 \in M}, (1-u_2)^{1_2 \in M}, ..., (1-u_n)^{1_n \in M}\right]; \tag{6}$$

where $N = \{1, 2, ..., n\}$, $m = |M|$ is the cardinal number of M, and $1_i \in M$ indicates the appartenence of i to M.

A n-dimensional stochastic process being a collection of random variables X_t; $t \in T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in $\mathbb{R}^d$, $n \in \mathbb{N}$, where T is the set of the parameters (space or time).

Let $C = \{C_{t_1,...,t_n}; t_1 < ... < t_n, n \in \mathbb{N}\}$ is a collection of copulas of stochastic process. satisfying the consistency condition

$$\lim_{u_k \to 1^-} C_{t\,_1,...,t\,_n}(u_1, ..., u_n) = C_{t\,_1,...,t\,_{k-1},t\,_{k+1}...,t\,_n}(u_1, ..., u_{k-1}, u_{k+1}, ..., u_n)$$

for all $u_i \in (0,1)$, $1 \leq k \leq n$ and $D = \{F_t, t \in T\}$ a collection of uni-dimensionnal distribution. Then, there exists a probability espace $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{(Y_t), t \in T\}$ such that

$$P\left(Y_{t_1} < x_1, ..., Y_{t_n} < x_n\right) = C_{t\,_1,...,t\,_n}\left(F_{t_1}(x_1), ..., F_{t_n}(x_n)\right). \tag{7}$$

for all $x_i \in \mathbb{R}$, $t_i \in T \subset \mathbb{R}$, $1 \leq i \leq n$ and Y_t is $\mathcal{F}_t$-mesurable for all $t \in T$.

Next, we present the notions of extremal processes which are processes obtained as limits of a normalized maxima processes of a sequence of random variables independent identically distributed (i.i.d) (or independent sequences not identically distributed). For more details, the reader can consult Resnick[19] and references therein.

Definition 2. *Suppose that $X_1, X_2, ...X_n$ is a sequence of independent and identically distributed random variables, and $M_n = max\{X_1, X_2, ..., X_n\}$.*
Let $\{Y(t), t \geq 0\}$ the stochastic process which is the natural limit when $n \to \infty$, of the process

$$Y_n(t) = (M_{[nt]} - a_n)/b_n, \qquad t \geq 1/n. \tag{8}$$

The joint distribution of vector Y_{t_1}, Y_{t_2},..., Y_{t_k} is same as the distribution of

$$U_1, max(U_1, U_2), ..., max(U_1, U_2, ..., U_k),$$

where U_1,U_2,...,U_k are independantes variables whose distribution is

$$F_{t_j - t_{j-1}} = G^{t_j - t_{j-1}},\ j = 1, ..., k \ with\ t_0 = 0. \tag{9}$$

where G is GEV type. Such a process is called extremal process.

The joint finite-dimensionnal distrbution is defined by

$$G_{t_1,t_2,...,t_k}(y_1, y_2, ..., y_k) = P[Y(t_1) < y_1, Y(t_2) < y_2, ..., Y(t_k) < y_k].$$

which gives, taking into account the independance

$$G_{t_1,t_2,...,t_k}(y_1, y_2, ..., y_k) = \left[G(\wedge_{i=1}^{k} y_i)\right]^{t_1} \times \left[G(\wedge_{i=2}^{k} y_i)\right]^{t_2-t_1} \times ... \times \left[G(y_k)\right]^{t_k - t_{k-1}}. \tag{10}$$

Let's remark that the advantage of these types of processes lies in the fact that they allow us to introduce a certain dynamic (evolution over time) in the modeling of extreme risks.

2.2. Overview of the tail multivariate dependence

The concept of tail dependence is widely used in multivariate analysis mainly in the bivariate case, see[10], [8] and [3]. However, their study in larger dimension is expanding. The generalization of tail dependence in dimension $d > 2$ consists in choosing $(h < d)$ variables and quantifying the conditional probability that each variables h take values in the tail knowing that the $d - h$ variables take this value too (see Barro [2]).

Let $X = (X_1, ..., X_d)$ be a random vector of $\mathbb{R}^d$ of joint distribution F and of copula C. We note, in the rest of this study, for all $h \leq d$, $X_{(h)} = (X_1, ..., X_h)$; $X_{(d-h)} = (X_{h+1}, ..., X_d)$; and by C_h and C_{d-h} their respective copulas. The generalization of the coefficients of upper tail dependencie $\lambda^{U,h}$, and lower tail dependencie $\lambda^{L,h}$, are given by:

$$\lambda^{U,h} = \lim_{u \to 1^-} P\{X_1 > F_1^{-1}(u), ..., X_h > F_h^{-1}(u)/X_{h+1} > F_{h+1}^{-1}(u), ..., X_d > F_d^{-1}(u)\}, \tag{11}$$

and

$$\lambda^{L,h} = \lim_{u \to 0^+} P\{X_1 \leq F_1^{-1}(u), ..., X_h \leq F_h^{-1}(u)/X_{h+1} \leq F_{h+1}^{-1}(u), ..., X_d \leq F_d^{-1}(u)\}. \tag{12}$$

In terms of copulas, (11) and (12) take respectively the form :

$$\lambda^{U,h} = \lim_{u \to 1^-} \frac{\bar{C}(1-u, ..., 1-u)}{\bar{C}_{d-h}(1-u, ..., 1-u)}, \text{ and } \lambda^{L,h} = \lim_{u \to 0^+} \frac{C(u, ..., u)}{C_{d-h}(u, ..., u)},$$

where C_{d-h} is the marginal copula of C associeted to vector $X_{(d-h)}$ and $\bar{C}$ is the survival copula associeted to C.

The four following sections present our main contribution in stochastic risk modeling.

3. Stochastic Risk modeling via extremal copulas

In this part of the paper, we propose some results on the multivariate extremal processes De-Haan [9] for more details. Let $Y(t) = \{(Y_1(t), ..., Y_d(t)\}$ be a vector of extremal process with joint distribution $F_t = (F_{1,t}, ..., F_{d,t})$, where $F_{i,t}$ is given by the formula (9). $Y(t)$ can be seen as the "limit" of $Y_n(t) = \{(Y_{1,n}(t), ..., Y_{d,n}(t))\}$ where $Y_{i,n}(t), i = 1, .., d$ are defined as in (8) :

$$P[Y_{1,n}(t) \leq y_1, ..., Y_{d,n}(t) \leq y_d] \to F_t(y_1, ..., y_d),\ n \to \infty. \tag{13}$$

and according to De-Haan[9], $F_t(y_1, ..., y_d) = G_*^t(y_1, ..., y_d)$ where G_* is the multivariate extreme value distribution.

Proposition 1. *Let $Y(t)$ be a vector of extremal processes in $\mathbb{R}^d$, with joint distribution F_t and marginal $F_{i,t}$, $i = 1, ..., d$. Then, there exists a unique convex function $B_t : [0, \infty] \times \mathcal{S}_d \to [0; \infty[$ such that a copula associeted to F_t be defined by,*

$$C_t^*(u_1, ..., u_d) = \exp\left\{-\left(\sum_{i=1}^{d} \tilde{u}_{i,t}\right) B_t\left(\frac{\tilde{u}_{1,t}}{\sum_{i=1}^{d} \tilde{u}_{i,t}}, ..., \frac{\tilde{u}_{d,t}}{\sum_{i=1}^{d} \tilde{u}_{i,t}}\right)\right\}, \tag{14}$$

where $\mathcal{S}_d = \{(x_1, .., x_d) \in \mathbb{R}^d / \sum_{i=1}^{d} x_i \leq 1\}$ is the simplex defined on $\mathbb{R}^d$ and $\tilde{u}_{i,t} = \mu_i - \frac{\sigma_i}{\xi_i}\left[1 - (-\frac{\ln u_i}{t})^{-\xi_i}\right],\ i = 1, 2, ..., d.$

Proof. According to Pickands, the function G_* is given (see Falk[12] and Resnick[19]) by:

$$G_*(x_1, ..., x_d) = \exp\left\{-\left(\sum_{i=1}^{d} x_i\right) A\left(\frac{x_1}{\sum_{i=1}^{d} x_i}, ..., \frac{x_d}{\sum_{i=1}^{d} x_i}\right)\right\}, \tag{15}$$

where the convex function $A : \mathcal{S}_d \to [\frac{1}{d}, 1]$, is the Pickands dependence function.

Since $F_t(x_1, ..., x_d) = G_*^t(x_1, x_2, ..., x_d)$, then

$$F_t(x_1, ..., x_d) = \exp\left\{-t\left(\sum_{i=1}^{d} x_i\right) A\left(\frac{x_1}{\sum_{i=1}^{d} x_i}, ..., \frac{x_d}{\sum_{i=1}^{d} x_i}\right)\right\} \tag{16}$$

the marginal distribution $F_{i,t}$ of F_t being continuous, according to Sklar's theorem, so, there exists a unique copula C_t^* such that,

$$F_t(x_1, ..., x_d) = C_t^*(F_{1,t}(x_1), ..., F_{d,t}(x_d)), \tag{17}$$

which can be write, by taking $u_i = F_{i,t}(x_i)$ for all i, such as

$$C_t^*(u_1, ..., u_d) = F_t(F_{1,t}^{-1}(u_1), ..., F_{d,t}^{-1}(u_d)). \tag{18}$$

Combining the relations (16) and (18) give us,

$$C_t^*(u_1, ..., u_d) = \exp\left\{-t\left(\sum_{i=1}^{d} F_{i,t}^{-1}(u_i)\right) A\left(\frac{F_{1,t}^{-1}(u_1)}{\sum_{i=1}^{d} F_{i,t}^{-1}(u_i)}, ..., \frac{F_{d,t}^{-1}(u_d)}{\sum_{i=1}^{d} F_{i,t}^{-1}(u_i)}\right)\right\},$$

which gives,

$$C_t^*(u_1, ..., u_d) = \exp\left\{-\left(\sum_{i=1}^{d} \tilde{u}_{i,t}\right) B_t\left(\frac{\tilde{u}_{1,t}}{\sum_{i=1}^{d} \tilde{u}_{i,t}}, ..., \frac{\tilde{u}_{d,t}}{\sum_{i=1}^{d} \tilde{u}_{i,t}}\right)\right\}, \tag{19}$$

with $B_t(x_1, ..., x_d) = tA(x_1, ..., x_d)$ and $\tilde{u}_{i,t} = F_{i,t}^{-1}(u_i)$, for all $i = 1, 2, .., d$ and $u_i \in [0, 1]$. The copula C_t^* being unique, so is the function B_t. The latter function is convex, being the product of a positive real and a convex function (function A)

The following result proposes a new measure of stochastic multivariate dependence.

Proposition 2. *Let $Y(t)$ be a vector of extremal processes in $\mathbb{R}^d$, with joint distribution F_t and marginal $F_{i,t}$, $i = 1, ..., d$. Let consider k random vector $\{Y(t_j) = (Y_{1,t_j}, ..., Y_{d,t_j}),\ j = 1, 2, .., k\}$ and $y^j = (y_1^j, ..., y_d^j) \in \mathbb{R}^d$. Then, the joint finite-dimensionnal copula $C^*_{t_1,t_2,...,t_k}$ of $Y(t_1), Y(t_2), ..., Y(t_k)$ is defined, for $0 = t_0 < t_1 < ... < t_k$ by:*

$$C^*_{t_1,t_2,...,t_k}[(u_1^j, ..., u_d^j)] = \exp\left\{-\sum_{m=1}^{k} L_{t_m}(\tilde{u}_{1,t_m}, \tilde{u}_{2,t_m}, ..., \tilde{u}_{d,t_m})\right\}; \tag{20}$$

where $(u_1^j, ..., u_d^j) \in [0; 1]^d$, for $j = 1; ..., k$ and L_{t_m} is a suitable convex funtion.

Proof. The joint finite-dimensionnal distribution of $\{Y(t_j) = (Y_{1,t_j}, Y_{2,t_j}, ..., Y_{d,t_j})\}$, for all $y^j = (y_j^1, ..., y_j^d)$ and $j = 1, 2, .., k$ is given by :

$$P[(Y_{1,t_j} \le y_1^j, Y_{2,t_j} \le y_2^j; ...; Y_{d,t_j} \le y_d^j)] = F_{t_1,t_2,...,t_k}((y_1^j, y_2^j, ..., y_d^j),$$

according to de Haan[9], Which gives

$$\begin{aligned} &P[(Y_{1,t_j} \le y_1^j; ...; Y_{d,t_j} \le y_d^j)] \\ &= F_{t_1}\left(\bigwedge_{j=1}^{k} y_1^j, ..., \bigwedge_{j=1}^{k} y_d^j\right) \times F_{t_2-t_1}\left(\bigwedge_{j=2}^{k} y_1^j; ...; \bigwedge_{j=2}^{k} y_d^j\right) \times ... \times F_{t_k-t_{k-1}}(y_1^k; ...; y_d^k) \end{aligned} \tag{21}$$

Let's take $u_i^j = F_{i,t_j}(y_i^j)$ for all $i = 1, ..., d$ and $j = 1, ..., k$, we have

$$C^*_{t_1,t_2,...,t_k}[(u_1^j, u_2^j, ..., u_d^j)] = F_{t_1,t_2,...,t_k}((F_{1,t_j}^{-1}(u_1^j), ..., F_{d,t_j}^{-1}(u_d^j)))$$

That leads to,

$$C^*_{t_1,t_2,...,t_k}[(u_1^j, ..., u_d^j)] \quad = \quad F_{t_1}\left(\bigwedge_{i=1}^{k} F_{t_1}^{-1}(u_1^i), \bigwedge_{i=1}^{k} F_{t_1}^{-1}(u_2^i), ..., \bigwedge_{i=1}^{k} F_{t_1}^{-1}(u_d^i)\right) \times \tag{22}$$

$$\times F_{t_2-t_2}\left(\bigwedge_{i=2}^{k} F^{-1}_{t_2-t_1}(u_1^i), \bigwedge_{i=2}^{k} F^{-1}_{t_2-t_1}(u_2^i), ..., \bigwedge_{i=2}^{k} F^{-1}_{t_2-t_1}(u_d^i)\right)$$
$$\times F_{t_k-t_{k-1}}\left(F^{-1}_{t_k-t_{k-1}}(u_1^k), F^{-1}_{t_k-t_{k-1}}(u_2^k), ..., F^{-1}_{t_k-t_{k-1}}(u_d^k)\right).$$

But according to Proposition 1, $C^*_{t_1,t_2,...,t_k}((u_1^j,...,u_d^j)$ for $j=1,...,k)$ can be written as

$$\begin{aligned}
C^*_{t_1,t_2,...,t_k}[(u_1^j,...,u_d^j)] &= \exp\left\{-\left(\sum_{i=1}^{d}\bigwedge_{j=1}^{k}\tilde{u}^j_{i,t_1}\right)B_{t_1}\left(\frac{\bigwedge_{j=1}^{k}\tilde{u}^j_{1,t_1}}{\sum_{i=1}^{d}\bigwedge_{j=1}^{k}\tilde{u}^j_{i,t_1}},...,\frac{\bigwedge_{j=1}^{k}\tilde{u}^j_{d-1,t_1}}{\sum_{i=1}^{d}\bigwedge_{j=1}^{k}\tilde{u}^j_{i,t_1}}\right)\right\}\times \\
&\times \exp\left\{-\left(\sum_{i=1}^{d}\bigwedge_{j=2}^{k}\tilde{u}^j_{i,t_2}\right)B_{t_2-t_1}\left(\frac{\bigwedge_{j=2}^{k}\tilde{u}^j_{1,t_2}}{\sum_{i=1}^{d}\bigwedge_{j=2}^{k}\tilde{u}^j_{i,t_2}},...,\frac{\bigwedge_{j=2}^{k}\tilde{u}^j_{d-1,t_2}}{\sum_{i=1}^{d}\bigwedge_{j=2}^{k}\tilde{u}^j_{i,t_2}}\right)\right\}\times \\
&\times \ ...\times\exp\left\{-\left(\sum_{i=1}^{d}\tilde{u}^k_{i,t_k}\right)B_{t_k-t_{k-1}}\left(\frac{\tilde{u}^k_{1,t_k}}{\sum_{i=1}^{d}\tilde{u}^k_{i,t_k}},...,\frac{\tilde{u}^k_{d-1,t_k}}{\sum_{i=1}^{d}\tilde{u}^k_{i,t_k}}\right)\right\}.
\end{aligned}$$

So, it comes that

$$C^*_{t_1,t_2,...,t_k}[(u_1^j,...,u_d^j)] = \exp\left\{-\sum_{m=1}^{k} L_{t_m}(\tilde{u}_{1,t_m},\tilde{u}_{2,t_m},...,\tilde{u}_{d,t_m})\right\} \tag{23}$$

where the dependence function L_{t_m} is given by

$$L_{t_m}(\tilde{u}_{1,t_m},...,\tilde{u}_{d,t_m}) = \left(\sum_{i=1}^{d}\bigwedge_{j=m}^{k}\tilde{u}^j_{i,t_j}\right)B_{t_m-t_{m-1}}\left(\frac{\bigwedge_{j=m}^{k}\tilde{u}^j_{1,t_j}}{\sum_{i=1}^{d}\bigwedge_{j=m}^{k}\tilde{u}^j_{i,t_j}},...,\frac{\bigwedge_{j=m}^{k}\tilde{u}^j_{d-1,t_j}}{\sum_{i=1}^{d}\bigwedge_{j=m}^{k}\tilde{u}^j_{i,t_j}}\right)$$

is a suitable convex function defined in the simplex of $\mathbb{R}^d$ while, for all $i=1,...,d$ and $j=1,...,k$, one have $\tilde{u}_{i,t_m}=\bigwedge_{j=m}^{k}\tilde{u}^j_{i,t_j}$, and

$$\tilde{u}^j_{i,t_j} = \mu_i - \frac{\sigma_i}{\xi_i}\left[1-\left(-\frac{\ln u_i^j}{t_j-t_{j-1}}\right)^{-\xi_i}\right]$$

4. The Expected Tail Dependence Coefficients

In this section, we first introduce the notion of expected tail dependence coefficient then we present essential properties related to the concept. The notion is a generalization of the notion of multivariate tail dependence coefficients (relations (11) and (12)) in the sense that they give us information on dependence average in the tails of multivariate distribution. In the bivariate case, it is close to the proposed risk measure by Brahim et al.[6].

Definition 3. *Let $X = (X_1, ..., X_d)$ a random vector in $\mathbb{R}^d$, with joint distribution F, marginal F_i, $i = 1, ..., d$ and copula C. We call the expected or average tail function of lower tail ζ_h^L and of upper tail ζ_h^U, the quantities respectively defined by:*

$$\zeta_h^L(u) = E\{X_{(h)}/X_{h+1} \leq F_{h+1}^{-1}(u), ..., X_d \leq F_d^{-1}(u)\},$$

respectively

$$\zeta_h^U(u) = E\{X_{(h)}/X_{h+1} > F_{h+1}^{-1}(u), ..., X_d > F_d^{-1}(u)\}.$$

A vectorial approach of these coefficients can be written:

$$\zeta_h^U(u) = \begin{pmatrix} E\{X_1/X_{h+1} > F_{h+1}^{-1}(u), ..., X_d > F_d^{-1}(u)\} \\ E\{X_2/X_{h+1} > F_{h+1}^{-1}(u), ..., X_d > F_d^{-1}(u)\} \\ ... \\ ... \\ E\{X_h/X_{h+1} > F_{h+1}^{-1}(u), ..., X_d > F_d^{-1}(u)\} \end{pmatrix};$$

for the lower coefficient, and

$$\zeta_h^L(u) = \begin{pmatrix} E\{X_1/X_{h+1} \leq F_{h+1}^{-1}(u), ..., X_d \leq F_d^{-1}(u)\} \\ E\{X_2/X_{h+1} \leq F_{h+1}^{-1}(u), ..., X_d \leq F_d^{-1}(u)\} \\ ... \\ ... \\ E\{X_h/X_{h+1} \leq F_{h+1}^{-1}(u), ..., X_d \leq F_d^{-1}(u)\} \end{pmatrix}.$$

for the right one.

It's abouts determining on average, having a vector $X = (X_1, ..., X_d)$, what is happening in the distribution tails of the $h \leq d$ random variables $X_1, ..., X_h$ knowing that the $(d-h)$ remaining variables each exceed its VaR at a threshold $u \in [0, 1]$ chosen.

Definition 4. *We call, respectively, expected (average) tail dependence of lower tail and upper tail respectively, the limits;*

$$\zeta_h^L = \lim_{u \to 0^+} \zeta_h^L(u), \textit{ and } \zeta_h^U = \lim_{u \to 1^-} \zeta_h^U(u). \tag{24}$$

The result below gives us the expressions of the marginal of the average dependence coefficients of lower and upper tail in terms of copulas.

Proposition 3. *Let $X = (X_1, ..., X_d)$ a random vector with copula C and suppose that the conditionnals densities $f_{X_i/X_{(d-h)}}$ and $\breve{f}_{X_i/X_{(d-h)}}$ of $\{X_i/X_{h+1}, ..., X_d\}$, respectively for lower and upper tail, exist. Then,*

i) the marginal expected tail dependence coefficient of lower tail of X is given by:

$$\zeta_{h,i}^L = \lim_{u \to 0^+} \int_u^1 VaR_{X_i}(\alpha).c_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha, \tag{25}$$

where

$c_{X_i/X_{(d-h)}}(\alpha,u,...,u) = \partial_{h+1,...,d}\dfrac{\partial_i C(u_i,u_{h+1},...,u_d)}{C_{d-h}(u_{h+1},...,u_d)}|_{(u_i,u_{h+1},...,u_d)=(\alpha,u,...,u)}$ *and*
$C(u_i,u_{h+1},...,u_d) = C(1,...,1,u_i,1,...1,u_{h+1},...,u_d)$.

ii) the marginal expected tail dependence coefficient of upper tail of X is given by:

$$\zeta^U_{h,i} = \lim_{u\to 1^-}\int_0^{1-u} VaR_{X_i}(1-\alpha).\check{c}_{X_i/X_{(d-h)}}(\alpha,u,...,u)d\alpha, \tag{26}$$

where

$\check{c}_{X_i/X_{(d-h)}}(1-\alpha,1-u,...,1-u) = \partial_{h+1,...,d}\dfrac{\partial_i\bar{C}(1-u_i,1-u_{h+1},...,1-u_d))}{\bar{C}_{d-h}(1-u_{h+1},...,1-u_d)}|_{(u_i,u_{h+1},...,u_d)=(\alpha,u,...,u)}$
and
$\bar{C}(1-u_i,1-u_{h+1},...,1-u_d) = \bar{C}(1,...,1,1-u_i,1,...1,1-u_{h+1},...,1-u_d)$.

Proof.

i) Each marginal coefficient for lower tail $\zeta^L_{h,i},\ i=1,..,h$ is defined by,

$$\zeta^L_{h,i} = \lim_{u\to 0^+} E\{X_i/X_{h+1} \le F^{-1}_{h+1}(u),...,X_d \le F^{-1}_d(u)\}$$

which gives

$$\zeta^L_{h,i} = \lim_{u\to 0^+}\int_{\phi_{X_i}(u)}^{+\infty} x f_{X_i/X_{(d-h)}}dx; \tag{27}$$

where $\phi_{X_i}(u)$ designates the u level quantile associeted to the variable X_i and $f_{X_i/X_{(d-h)}}$ is the conditional density associeted to $\{X_i/X_{h+1},...,X_d\}$, that we suppose the existence.
The associeted distribution function $F_{X_i/X_{(d-h)}}$ of $f_{X_i/X_{(d-h)}}$ is equal to,

$$F_{X_i/X_{(d-h)}}(x_i,x_{h+1},...,x_d) = P[X_i \le x_i/X_{h+1} \le x_{h+1},...,X_d \le x_d]$$

Furthermore, we have

$$F_{X_i/X_{(d-h)}}(x_i,x_{h+1},...,x_d) = \frac{P[X_i \le x_i, X_{h+1} \le x_{h+1},...,X_d \le x_d]}{P[X_{h+1} \le x_{h+1},...,X_d \le x_d]}$$

and by using Sklar's theorem, it follows that

$$F_{X_i/X_{(d-h)}}(x_i,x_{h+1},...,x_d) = \frac{C(F_i(x_i),F_{h+1}(x_{h+1}),...,F_d(x_d))}{C_{d-h}(F_{h+1}(x_{h+1}),...,F_d(x_d))} \tag{28}$$

where

$$C(F_i(x_i),F_{h+1}(x_{h+1}),...,F_d(x_d)) = C(1,1,...1,F_i(x_i),1,..1,F_{h+1}(x_{h+1}),...,F_d(x_d)).$$

Then the corresponding density $f_{X_i/X_{(d-h)}}$ is given by,

$$f_{X_i/X_{(d-h)}}(x_i, x_{h+1}, ..., x_d) = \partial_{i,h+1,...,d} \frac{C(F_i(x_i), F_{h+1}(x_{h+1}), ..., F_d(x_d))}{C_{d-h}(F_{h+1}(x_{h+1}), ..., F_d(x_d))}.$$

which gives

$$f_{X_i/X_{(d-h)}}(x_i, x_{h+1}, ..., x_d) = f_i(x_i)\partial_{h+1,...,d} \left(\frac{\partial_i C(F_i(x_i), F_{h+1}(x_{h+1}), ..., F_d(x_d))}{C_{d-h}(F_{h+1}(x_{h+1}), ..., F_d(x_d))} \right) \tag{29}$$

for all $(u_i, u_{h+1}, ..., u_d) \in [0, 1] \times [0, 1]^{d-h}$,

$$c_{X_i/X_{(d-h)}}(u_i, u_{h+1}, ..., u_d) = \partial_{h+1,...,d} \frac{\partial_i C(u_i, u_{h+1}, ..., u_d)}{C_{d-h}(u_{h+1}, ..., u_d)}$$

it follows that,

$$c_{X_i/X_{(d-h)}}(\alpha, u, ..., u) = \partial_{h+1,...,d} \frac{\partial_i C(u_i, u_{h+1}, ..., u_d)}{C_{d-h}(u_{h+1}, ..., u_d)} |_{(u_i,u_{h+1},...,u_d)=(\alpha,u,...,u)}$$

where $C(u_i, u_{h+1}, ..., u_d) = C(1, ...1, u_i, 1...1, u_{h+1}, ..., u_d)$.

Finally,

$$\zeta^L_{h,i} = \lim_{u\to 0^+} \int_u^1 VaR_{X_i}(\alpha)c_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha.$$

ii) In the same way, for each marginal coefficient of upper tail $\zeta^L_{h,i}$, $i = 1, .., h$; the conditional distribution of $\{X_i > ./X_{h+1} > ., ..., X_d > .\}$ is given by;

$$\breve{F}(x_i, x_{h+1}, ..., x_d) = \frac{\bar{C}(1 - F_i(x_i), 1 - F_{h+1}(x_{h+1}), ..., F_d(x_d))}{\bar{C}_{d-h}(1 - F_{h+1}(x_{h+1}), ..., F_d(x_d))}$$

so, the corresponding density $\breve{f}_{X_i/X_{(d-h)}}$ is obtained,

$$\breve{f}_{X_i/X_{(d-h)}}(x_i, x_{h+1}, ..., x_d) = -f_i(x_i)\partial_{h+1,...,d} \frac{\partial_i \bar{C}_{d-h}(1 - F_i(x_i), 1 - F_{h+1}(x_{h+1}), ..., F_d(x_d))}{\bar{C}(1 - F_{h+1}(x_{h+1}), ..., F_d(x_d))},$$

it follows that, for all $(u_i, u_{h+1}, ..., u_d) \in [0, 1] \times [0, 1]^{d-h}$,

$$\breve{c}_{X_i/X_{(d-h)}}(u_i, u_{h+1}, ..., u_d) = \partial_{h+1,...,d} \frac{\partial_i \bar{C}(1 - u_i, 1 - u_{h+1}, ..., 1 - u_d))}{\bar{C}_{d-h}(1 - u_{h+1}, ..., 1 - u_d)},$$

which implies,

$$\breve{c}_{X_i/X_{(d-h)}}(\alpha, u, ..., u) = \partial_{h+1,...,d} \frac{\partial_i \bar{C}(1 - u_i, 1 - u_{h+1}, ..., 1 - u_d))}{\bar{C}_{d-h}(1 - u_{h+1}, ..., 1 - u_d)} |_{(u_i,u_{h+1},...,u_d)=(\alpha,u,...,u)},$$

with $\bar{C}(1 - u_i, 1 - u_{h+1}, ..., 1 - u_d) = \bar{C}(1, ...1, 1 - u_i, 1...1, 1 - u_{h+1}, ..., 1 - u_d)$.

and finaly,

$$\zeta^U_{h,i} = \lim_{u\to 1^-} \int_0^{1-u} VaR_{X_i}(1 - \alpha)\breve{c}_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha.$$

It is easy to remark that for the particular case where the copula associeted to the vector $X = (X_1, ..., X_d)$ is the independant one, that is : $C(u_1, ..., u_n) = u_1 \times ... \times u_n$, then,

$$C_{X_i/X_{(d-h)}}(u_i, u_{h+1}, ..., u_n) = u_i, \quad \Rightarrow \quad c_{X_i/X_{(d-h)}}(u_i, u_{h+1}, ..., u_n) = 0$$

and

$$\breve{C}_{X_i/X_{(d-h)}}(u_i, u_{h+1}, ..., u_n) = 1 - u_i, \quad \Rightarrow \quad \breve{c}_{X_i/X_{(d-h)}}(u_i, u_{h+1}, ..., u_n) = 0$$

from where $\zeta_h^U = 0$ and $\zeta_h^L = 0$.

Belonging both to the max-stable and the Archimax families, the logistic family plays a key role in extremal modeling.

Corollary 1. *The classic copula associated to the logistic family (Gumbel's copula) does not admit coefficient of average dependence of lower tail, while that of upper tail is asymptotically negligible.*

Proof. Let (X, Y) a bivariate vector with joint distribution $F(x, y) = (1-e^{-x}-e^{-y})^{-1}$. The marginal distributions are equal to $F_X(t) = F_Y(t) = (1 - e^{-t})^{-1}$ and the associeted copula is $C(u, v) = \dfrac{uv}{u + v - uv}$. The conditional copula for lower tail $C_{X/Y}$ is given by

$$C_{X/Y}(u, v) = \frac{u}{u + v - uv},$$

i) For the upper tail $\breve{C}_{X/Y}$ it comes that

$$\breve{C}_{X/Y}(u, v) = \frac{(1 - u - v)(u + v - uv) + uv}{(1 - v)(u + v - uv)} = 1 - \frac{u^2}{u + v - uv}.$$

The average dependence functions of lower and upper tail are calculated such as:

$$\zeta^L(v) = \int_v^1 \left[\frac{x - v + xv}{(x + v - xv)^3} \ln\left(\frac{x}{1 - x}\right)\right] dx.$$

Dealing with integration once by parts, it follows that

$$\zeta^L(v) = \frac{1 - v}{(2 - v)^2} \ln(\frac{v}{1 - v}) - \int_v^1 \frac{1}{x(1 - x)} \frac{-x(1 - x)}{(x + v - xv)^2} dx$$

which gives, twice by parts

$$\zeta^L(v) = \frac{(1 - v)}{(2 - v)^2} \ln(\frac{v}{1 - v}) + \frac{1 - v}{2v - v^2} \tag{30}$$

and finally, we have

$$\zeta^L = \lim_{v \to 0^+} \frac{(1 - v)}{(2 - v)^2} \ln\left(\frac{v}{1 - v}\right) + \frac{1 - v}{2v - v^2} = +\infty.$$

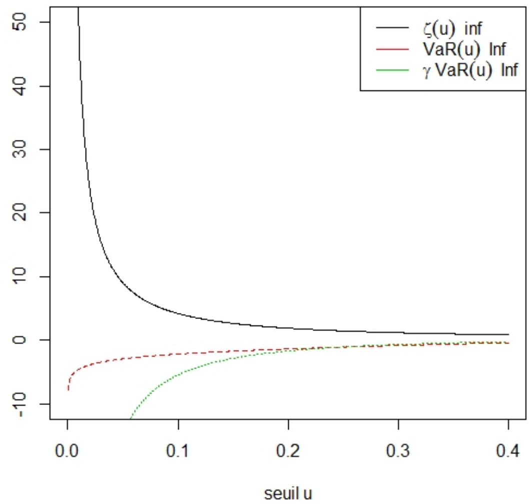

Figure 1: Expected (average) dependence function of lower tail and the VaR of lower tail for the copula associated with the bivariate logistics distribution of Gumbel.

The figure1, gives us the evolution of the average dependence function of the lower tail, with the corresponding VaR and the bound defined in relation (52).

ii) For the upper tail, it comes that

$$\zeta^U(v) = \int_0^v \left[\frac{-x^4(1-v) - xv(-x^2-2x+2)}{(x+v-xv)^3}\ln\left(\frac{x}{1-x}\right)\right]dx$$

which gives, once by parts

$$\zeta^U(v) = \ln(\frac{v}{1-v})\frac{v^2(1-v)}{(2v-v^2)} - \int_0^v \frac{x}{(x+v-xv)^2}dx$$

and twice by parts

$$\zeta^U(v) = \frac{v(1-v)}{(2-v)}\ln(\frac{v}{1-v}) - \frac{1-v}{2-v} + \int_0^v \frac{1-x}{x(1-v)+v}dx$$

By taking

$$R(v) = \frac{v(1-v)}{(2-v)}\ln(\frac{v}{1-v}) - \frac{1-v}{2-v}$$

one obtains

$$\zeta^U(v) = R(v) + \left[\frac{1-x}{1-v}\ln(x(1-v)+v)\right]_0^v - \frac{1}{1-v}\int_0^v \ln(x(1-v)+v)dx$$

So, it comes that:

$$\zeta^U(v) = R(v) + \ln(v(1-v)+v) - \frac{1}{1-v}\left(\left[\frac{x(1-v)+v}{1-v}\ln(x(1-v)+v)\right]_0^v - \int dx\right).$$

And finally:

$$\zeta^U(v) = \frac{v(1-v)}{(2-v)}\ln\left(\frac{v}{1-v}\right) + \left(-\frac{1-v}{2-v} + \frac{2v-v^2}{(1-v)^2}\right)\ln(2v-v^2) + 1. \tag{31}$$

Therefore, it comes that

$$\zeta^U = \lim_{u\to 1^-}\frac{v(1-v)}{(2-v)}\ln\left(\frac{v}{1-v}\right) + \left(-\frac{1-v}{2-v} + \frac{2v-v^2}{(1-v)^2}\right)\ln(2v-v^2) + 1 = 0.$$

The figure2, gives us the evolution of the expected dependence function of upper tail, with the corresponding VaR and the bound defined in relation (53).

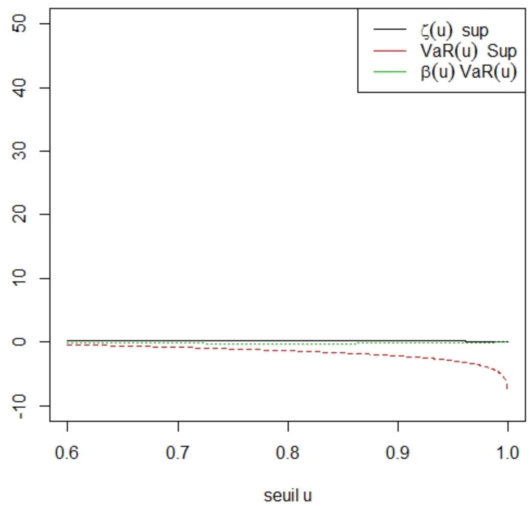

Figure 2: Expected dependence function of the upper tail and VaR of upper tail of the copula associated with the bivariate logistics distribution of Gumbel.

Corollary 2. *Let (X,Y) be a bivariate random variable which marginal distribution is standard Fréchet one $F_X(x) = \exp\{-x^{-\alpha}\}$ and that the copula which determines their dependence structure is Gumbel-Hougaard bivariate copula. Then,*

$$\zeta^L = +\infty, \text{ and } \zeta^U = +\infty. \tag{32}$$

Proof. The bivariate Gumbel-Hougaard copula (see [18]) is defined $\forall u, v \in [0,1]$ and $\theta \geq 1$, by

$$C_\theta(u,v) = \exp\Big\{ - \big[(-\ln(u))^\theta + (-\ln(v)))\theta\big]^{1/\theta}\Big\}. \tag{33}$$

The conditional copulas $C_{X/Y}$ and $\breve{C}_{X/Y}$ are given by

$$C_{X/Y}(u,v) = \frac{C_\theta(u,v)}{v}. \tag{34}$$

and

$$\breve{C}_{X/Y}(u,v) = \frac{1-u-v+C_\theta(u,v)}{1-v}. \tag{35}$$

So, we obtain the densities $c_{X/Y}$ and $\breve{c}_{X/Y}$, respectively by

$$\begin{aligned}\breve{c}_{X/Y}(u,v) \quad &= \quad \frac{C_\theta(u,v)}{uv(1-v)}\Bigg[(1-\theta)\Big[(-\ln u)^\theta + (-\ln v)^\theta\Big]\Big[^{\frac{1-2\theta}{\theta}}(-\ln u)(-\ln v)\Big]^{\theta-1} + \\ &+ \left[\begin{array}{c}\left[(-\ln v)\left[^{\theta-1}(-\ln u)^\theta + (-\ln v)^\theta\right]^{\frac{1-\theta}{\theta}} + \dfrac{v}{1-v}\right]\times \\ \left[(-\ln u)^{\theta-1}\left[(-\ln u)^\theta + (-\ln v)^\theta\right]\right.\end{array}\right].\end{aligned} \tag{36}$$

The expected tail dependence functions are calculated, $\zeta^L(u) = \int_u^1 (-\ln t)^{-1/\alpha} c_{X/Y}(t,u)dt$ which gives :

$$\begin{aligned}\zeta^L(u) \quad &= \quad \left[-(-\ln t)^{-1/\alpha}\frac{C_\theta(t,u)}{tu^2}\Big[(-\ln u)^{\theta-1}(-\ln t)^\theta + (-\ln u)^\theta\Big]^{\frac{1-\theta}{\theta}} + 1\right]_u^1 + \\ &+ \int_u^1 \left[\frac{(-\ln t)^{\frac{-1-\alpha}{\alpha}}}{\alpha}\frac{C_\theta(t,u)}{tu^2}\Big[(-\ln u)^{\theta-1}(-\ln t)^\theta + (-\ln u)^\theta\Big]^{\frac{1-\theta}{\theta}} + 1\right] dt\end{aligned}$$

and finally

$$\zeta^L(u) = (-\ln u)^{-1/\alpha} u^{(2^{1/\theta}-2)}(2^{\frac{1-\theta}{\theta}} + 1) + \frac{(-\ln u)^{\theta-1}}{\alpha u^2} K_1(u), \tag{37}$$

where

$$K_1(u) = \int_u^1 \left[(-\ln t)^{\frac{-1-\alpha}{\alpha}}\frac{C_\theta(t,u)}{t}\Big[\Big[(-\ln t)^\theta + (-\ln u)^\theta\Big]^{\frac{1-\theta}{\theta}} + (-\ln u)^{1-\theta}\Big]\right] dt.$$

The figure 3 shows the behavior of average dependence function of lower tail for different values of the parameter ($\theta = 1.5$ and $\theta = 1.7$) of the copula. We see that it is a increasing function of the parameter.

$$\zeta^U(u) = \int_0^u (-\ln(1-t))^{-1/\alpha}\breve{c}_{X/Y}(t,u)dt$$

which gives

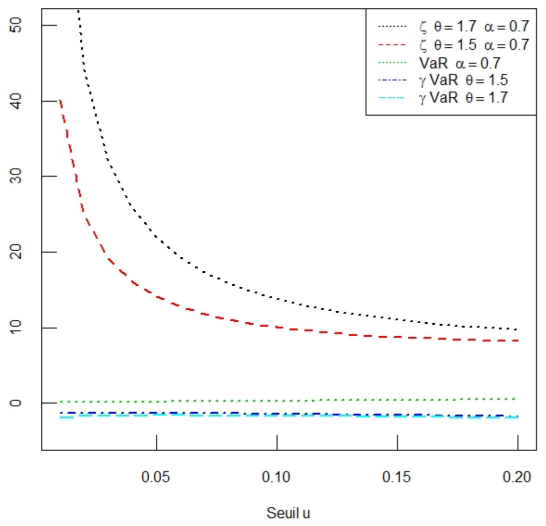

Figure 3: Lower tail average dependence function and lower tail VaR for the Gumbel-Hougaard copula with standard frechet margin with parameter α.

$$\zeta^U(u) = \Bigg[(-\ln(1-t))^{-1/\alpha}\frac{C_\theta(t,u)}{(1-u)^2}\Bigg[\frac{1-u}{u}\Big\{-(-\ln u)^{\theta-1}\big[(-\ln t)^\theta+(-\ln u)^\theta\big]^{\frac{1-\theta}{\theta}}\Big\}+1\Bigg]\Bigg]_0^u + $$
$$+ \int_0^u \Bigg[\frac{(-\ln(1-t))^{\frac{-1-\alpha}{\alpha}}}{\alpha t}\frac{C_\theta(t,u)}{(1-u)^2}\Bigg[\frac{1-u}{u}\Big\{-(-\ln u)^{\theta-1}\big[(-\ln t)^\theta+(-\ln u)^\theta\big]^{\frac{1-\theta}{\theta}}\Big\}+1\Bigg]\Bigg]dt$$

and finally

$$\zeta^U(u) = \Big[\frac{1-u}{u}2^{(\frac{1-\theta}{\theta})}+1\Big]\frac{u^{2^{1/\theta}}(-\ln(1-u))^{-1/\alpha}}{(1-u)^2} + \frac{(-\ln u)^{\theta-1}}{\alpha u(1-u)}K_2(u), \qquad (38)$$

where
$K_2(u) = \int_0^u \frac{(-\ln(1-t))^{\frac{-1-\alpha}{\alpha}}}{t}C_\theta(t,u)\Big[\Big\{-\big[(-\ln t)^\theta+(-\ln u)^\theta\big]^{\frac{1-\theta}{\theta}}\Big\}+\frac{u}{(1-u)(-\ln u)^{\theta-1}}\Big]dt.$

The figure 4 shows the behavior of average dependence function of upper tail for different values of the parameter ($\theta = 1.5$ and $\theta = 1.7$) of the copula. We see that it's a increasing function of the parameter.

5. Density of Conditional Extremal copulas

The resultat below gives us the form of the density $c_{X_i/X_{(d-h)}}$ in the case of conditional extremal copula.

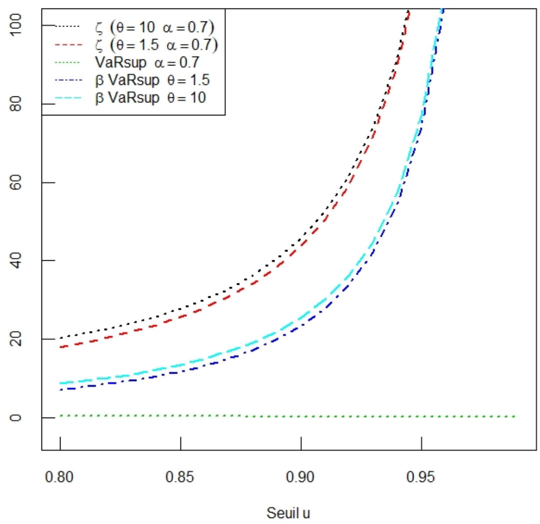

Figure 4: Lower tail average dependence function and upperr tail VaR for the Gumbel-Hougaard copula with standard frechet margin with parameter α.

Proposition 4. *Let $X = (X_1, ..., X_d)$ a random vector with extremal copula C. Let note $u = (u_i, u_{h+1}, ..., u_d) \in [0,1] \times [0,1]^{d-h}$ and l stable tail dependence function. Then the density function $c_{X_i/X_{(d-h)}}$ of the copula $C_{X_i/X_{(d-h)}}$ associeted to the conditionnal vector $\{X_i/X_{(d-h)}\}$ is given by;*

$$c_{X_i/X_{(d-h)}}(u) = \frac{C_{X_i/X_{(d-h)}}(u)}{u_i \prod_{h+1}^{d} u_j} \left[\sum_{\kappa \in \mathcal{E}} \partial_\kappa \left(\partial_i l(\tilde{u}) \right) . \left(\sum_{\pi \in \Pi} (-1)^{|\pi|} \prod_{B \in \pi} \partial_B L(\tilde{u}) \right) \right] \tag{39}$$

where $\tilde{u} = -\log(u)$; $\partial_i(.)$ indicates the partial derivative with respect to the i-th variable; $\prod u = u_i.u_{h+1}...u_d$, $\mathcal{E}$ is the set of all parts of $E = \{h+1, h+2, ..., d\}$, π runs through the set Π of partitions of κ^c, the complement of κ in E, $B \in \pi$ significates that B runs through the set π; $|\pi|$ designates the cardinal number of π and $L(x_i, x_{h+1}, ..., x_d) = l(x_i, x_{h+1}, ..., x_d) - l_{d-h}(x_{h+1}, ..., x_d)$.

For prove Proposition 10 we need the following Lemma which establishes a property of two partially devative functions.

Lemma 1. *Let f and g two functions defined on $\mathbb{R}^d$ such that their partial derivatives of all order exist and are continuous. Then, for all $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$*

$$\partial_{1,2,...,d}(f(x).g(x)) = \sum_{\kappa \in \mathcal{E}} \partial_\kappa f(x).\partial_{\kappa^c} g(x), \tag{40}$$

where $\mathcal{E}$ is the set of all parts of $E = \{1, 2, ..., d\}$ and κ^c is the complement of κ in E.

The proof of this lemma is given by induction for recurrence relation.
Proof.

i) At first order, we have :
If $E = \{1\}$ then $\mathcal{E} = \{\emptyset, \{1\}\}$ therefore $\kappa = \emptyset$ or $\kappa = \{1\}$,

$$\partial_1(f(x).g(x)) = \sum_{\kappa=\emptyset\,,\{1\}} \partial_\kappa f(x).\partial_{\kappa^c} g(x) = \partial_\emptyset f(x).\partial_{\{1\}} g(x) + \partial_{\{1\}} f(x).\partial_\emptyset g(x)$$

which gives

$$\partial_1(f(x).g(x)) = f(x)\partial_1 g(x) + g(x)\partial_1 f(x).$$

So, the formula is true at first order.

ii) Suppose that the formula is true at order $for\ p \geq 1$, that is

$$\partial_{1,2,\ldots,p}(f(x).g(x)) = \sum_{\kappa\in\mathcal{E}} \partial_\kappa f(x).\partial_{\kappa^c} g(x), \tag{41}$$

where $\mathcal{E}$ is the set of all parts of $E = \{1, 2, ..., p\}$ and κ^c is the complement of κ in E.

iii) Let's prove that the formula is true at order $p + 1$,

$$\begin{aligned}
\partial_{1,2,\ldots,p,p+1}(f(x).g(x)) &= \partial_{p+1}(\partial_{1,2,\ldots,p}(f(x).g(x))) \\
&= \partial_{p+1}\left(\sum_{\kappa\in\mathcal{E}} \partial_\kappa f(x).\partial_{\kappa^c} g(x)\right) \\
&= \sum_{\kappa\in\mathcal{E}} \partial_{p+1}\big(\partial_\kappa f(x).\partial_{\kappa^c} g(x)\big) \\
&= \sum_{\kappa\in\mathcal{E}} \left[\sum_{\kappa'=\emptyset\,,\{p+1\}} \partial_{\kappa'}\big[\partial_\kappa f(x)\big].\partial_{\kappa'^c}\big[\partial_{\kappa^c} g(x)\big]\right] \\
&= \sum_{\kappa\in\mathcal{E}} \sum_{\kappa'=\emptyset\,,\{p+1\}} \partial_{\kappa'\cup\kappa} f(x).\partial_{\kappa'^c\cup\kappa^c} g(x). \qquad (42)
\end{aligned}$$

So, finally, we get

$$\partial_{1,2,\ldots,p,p+1}(f(x).g(x)) = \sum_{\kappa_1\in\mathcal{E}_1} \partial_{\kappa_1} f(x).\partial_{\kappa_1^c} g(x). \tag{43}$$

In (42), note that for $\kappa' = \emptyset\,, \{p+1\}$ ($\kappa'^c = \{p+1\}\,, \emptyset$) and $\forall\kappa \in \mathcal{E}$ the set of parts of E, then:

$$\begin{aligned}
(\kappa' \cup \kappa) \cap (\kappa'^c \cup \kappa^c) &= \emptyset \\
(\kappa' \cup \kappa) \cup (\kappa'^c \cup \kappa^c) &= \{1, 2, ..., p, p+1\}, \qquad (44)
\end{aligned}$$

and also $\kappa'\cup\kappa$ runs through the set $\mathcal{E}_1$ of all parts of $E_1 = \{1, 2, ..., p, p+1\}$. Therefore (40) holds

Now, we are able to proove Proposition 4

Proof. (of Proposition 4) Since the copula C is extreme, then it have the representation(see Falk[12]):

$$C(u_1, u_2, ..., u_d) = \exp\big\{ -l(-\log u_1, -\log u_2, ..., -\log u_d)\big\}, \tag{45}$$

where the function $l(.)$ is the stable tail dependence function associeted to C.

Then, for $u = (u_i, u_{h+1}, ..., u_d) \in [0,1] \times [0,1]^{d-h}$,

$$C_{X_i/X_{(d-h)}}(u_i, u_{h+1}, ..., u_d) = \frac{\exp\{-l(-\log u_i, -\log u_{h+1}, ..., -\log u_d)\}}{\exp\{-l_{d-h}(-\log u_{h+1}, ..., -\log u_d)\}}.$$

Then the density is given by,

$$c_{X_i/X_{d-h}}(u_i, u_{h+1}, ..., u_d) = \partial_{h+1,...,d}\frac{\partial_i \exp\big\{ -l(-\log u_i, -\log u_{h+1}, ..., -\log u_d)\big\}}{\exp\big\{ -l_{d-h}(-\log u_{h+1}, ..., -\log u_d)\big\}}.$$

That gives

$$c_{X_i/X_{d-h}}(u_i, u_{h+1}, ..., u_d) = \frac{1}{u_i}\partial_{h+1,...,d}\Big[\partial_{u_i} l(-\log u). \exp\{-L(-\log u)\}\Big] \tag{46}$$

That is :

$$c_{X_i/X_{d-h}}(u_i, u_{h+1}, ..., u_d) = \frac{1}{u_i}\sum_{\kappa\in\mathcal{E}} \partial_\kappa\big(\partial_{u_i} l(-\log u)\big).\partial_{\kappa^c} \exp\{-L(-\log u)\}. \tag{47}$$

with

$$L(x_i, x_{h+1}, ..., x_d) = l(x_i, x_{h+1}, ..., x_d) - l_{d-h}(x_{h+1}, ..., x_d)$$

; $\mathcal{E}$ being the set of all parts of $E = \{h+1, ..., d\}$ and κ^c the complement of κ in E.

Using the formula of Fàa di Bruno, it follows for all $x = (x_1, ..., x_d) \in \mathbb{R}^d$;

$$\frac{\partial^d}{\partial x_1 ... \partial x_d} f(g(x)) = \sum_{\pi\in\Pi} f^{(|\pi|)}(g(x)). \prod_{B\in\pi} \frac{\partial^{(|B|)} g(x)}{\prod_{j\in B}\partial x_j}; \tag{48}$$

where π runs through Π the set of partitions of $\{1, 2, ..., d\}$ and $B \in \pi$ significates that it runs through the elements of π.

Furthermore, by taking $x = -\log(u)$, one can remark that the expression $\exp\{-L(x)\}$ is of the form $f(g(x))$. Then the expression

$$\partial_{\kappa^c} \exp\{-L(-\log u)\} = \sum_{\pi\in\Pi} (-1)^{(|\pi|)} \exp\{-L(x)\}. \prod_{B\in\pi} \frac{\partial^{(|B|)} L(x)}{\prod_{j\in B}\partial u_j};$$

where Π is the set of partitions of κ^c.

By noticing that

$$t\frac{\partial^{(|B|)} L(x)}{\prod_{j\in B}\partial u_j} = \frac{\partial^{(|B|)} L(x)}{\prod_{j\in B}\partial x_j}. \prod_{j\in B}\frac{\partial x_j}{\partial u_j},$$

on obtains,

$$\exp\{-L(x)\}\sum_{\pi\in\Pi}(-1)^{|\pi|}\prod_{B\in\pi}\frac{\partial^{(|B|)}L(x)}{\prod_{j\in B}\partial x_j}.\prod_{j\in B}\frac{\partial x_j}{\partial u_j}=\exp\{-L(x)\}\sum_{\pi\in\Pi}(-1)^{(|\pi|)}\prod_{B\in\pi}\left(\frac{\partial^{(|B|)}L(x)}{\prod_{j\in B}\partial x_j}.\prod_{j\in B}\frac{-1}{u_j}\right).$$

which gives

$$\exp\{-L(x)\}\sum_{\pi\in\Pi}(-1)^{|\pi|}\prod_{B\in\pi}\frac{\partial^{(|B|)}L(x)}{\prod_{j\in B}\partial x_j}.\prod_{j\in B}\frac{\partial x_j}{\partial u_j}=\exp\{-L(x)\}\sum_{\pi\in\Pi}(-1)^{|\pi|}\left(\prod_{B\in\pi}\prod_{j\in B}\frac{-1}{u_j}\right)\prod_{B\in\pi}\frac{\partial^{(|B|)}L(x)}{\prod_{j\in B}\partial x_j}.$$

And finally,

$$\exp\{-L(x)\}\sum_{\pi\in\Pi}(-1)^{|\pi|}\prod_{B\in\pi}\frac{\partial^{(|B|)}L(x)}{\prod_{j\in B}\partial x_j}.\prod_{j\in B}\frac{\partial x_j}{\partial u_j}=\frac{-\exp\{-L(x)\}}{\prod_{j\in\kappa^c}U_j}\sum_{\pi\in\Pi}(-1)^{|\pi|}\prod_{B\in\pi}\partial_B L(x). \tag{49}$$

In other hand, we have

$$\partial_\kappa\big(\partial_i l(-\log u)\big)=\frac{\partial^{(|\kappa|)}(\partial_i l(x))}{\prod_{j\in\kappa}\partial x_j}.\frac{-1}{\prod_{j\in\kappa}u_j}$$

That is:

$$\partial_\kappa\big(\partial_i l(-\log u)\big)=\frac{-1}{\prod_{j\in\kappa}u_j}.\partial_\kappa(\partial_i l(x)) \tag{50}$$

by using the relations (49) and (50), we obtain (39) as disserted.

6. A relationship between VaR and TVaR

It is convenient in risk analysis to compare each risk measure to VaR (which is a reference) or to its derivative measures, the tail value-at-risk of risk X at the $\alpha\in[0,1]$ level defined by,

$$TVaR(\alpha)=\frac{1}{1-\alpha}\int_\alpha^1 VaR_X(t)dt. \tag{51}$$

For detailed statements , see [16].

Corollary 3. *Let $X=(X_1,...,X_d)$ be a random vector with extremal copula C . The marginal of the expected tail dependence is defined, for all $u\in[0,1]$, by*

$$\zeta^L_{h,i}=\lim_{u\to 0^+}\frac{1}{(u)^{d-h}}\int_u^1\frac{VaR_{X_i}(\alpha)}{\alpha}C_{X_i/X_{(d-h)}}(\alpha,u,...,u)H_i(\alpha,u)d\alpha,$$

where

$$H_i(\alpha,u)=\Big\lfloor\textstyle\sum_{\kappa\in\mathcal{E}}\partial_\kappa\big(\partial_i l(x)\big).\big(\sum_{\pi\in\Pi}(-1)^{|\pi|}\prod_{B\in\pi}\partial_B L(x)\big)\Big\rfloor_{x=(-\log\alpha,-\log u,-\log u...,-\log u)}.$$

Proof. This result is immediat by using relations (25) and (39) and taking $u_i = \alpha, u_{h+1} = u, ..., u_d = u$ for all $i = 1, ..., h$ and $h \leq d$.

Corollary 4. *Let $X = (X_1, ..., X_d)$ be a random vector, with distribution F and associeted copula C. for all $i = 1, ..., h$ with $h \leq d$, then*

i) the marginal of the expected tail dependence function of lower tail verify, for all u near 0

$$\zeta^L_{h,i}(u) \geq \gamma(u).VaR_{X_i}(u), \tag{52}$$

where $\gamma(u) = \int_u^1 c_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha$.

ii) the marginal of the expected tail dependence function of upper tail verify, for all u near 1

$$\zeta^U_{h,i}(u) \geq \beta(u).VaR_{X_i}(1-u), \tag{53}$$

where $\beta(u) = \int_0^u \breve{c}_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha$.

Proof. For all $u, \alpha \in [0, 1[$ such that $u \leq \alpha$, it follows that

$$VaR_{X_i}(\alpha) \geq VaR_{X_i}(u), \tag{54}$$

since $\partial_{h+1,...,d}\dfrac{\partial_i C(u_i, u_{h+1}, ..., u_d)}{C_{d-h}(u_{h+1}, ..., u_d)}$ is a density, we have

$$\partial_{h+1,...,d}\frac{\partial_i C(u_i, u_{h+1}, ..., u_d)}{C_{d-h}(u_{h+1}, ..., u_d)} \geq 0, \tag{55}$$

which implies,

$$c_{X_i/X_{(d-h)}}(\alpha, u, ..., u) \geq 0, \tag{56}$$

so,

$$VaR_{X_i}(\alpha)c_{X_i/X_{(d-h)}}(\alpha, u, ..., u) \geq VaR_{X_i}(u)c_{X_i/X_{(d-h)}}(\alpha, u, ..., u).$$

Moreover,

$$\begin{aligned} \zeta^L_{h,i}(u) &= \int_u^1 VaR_{X_i}(\alpha)c_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha \\ &\geq VaR_{X_i}(u)\int_u^1 c_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha \\ &= VaR_{X_i}(u).\gamma(u). \end{aligned} \tag{57}$$

Finally,

$$\zeta^L_{h,i}(u) \geq \gamma(u).VaR_{X_i}(u), \;\; i = 1, .., h. \tag{58}$$

For all $u, \alpha \in [0, 1[$ such that $\alpha \leq u$, then

$$VaR_{X_i}(1-\alpha) \geq VaR_{X_i}(1-u), \tag{59}$$

Since $\partial_{h+1,...,d}\dfrac{\partial_i\bar{C}(1-u_i,1-u_{h+1},...,1-u_d))}{\bar{C}_{d-h}(1-u_{h+1},...,1-u_d)}$ is a density function, it follows that

$$\partial_{h+1,...,d}\frac{\partial_i\bar{C}(1-u_i,1-u_{h+1},...,1-u_d))}{\bar{C}_{d-h}(1-u_{h+1},...,1-u_d)} \geq 0, \tag{60}$$

which implies,

$$\breve{c}_{X_i/X_{(d-h)}}(\alpha,u,...,u) \geq 0, \tag{61}$$

Finally,

$$\zeta^U_{h,i}(u) \geq VaR_{X_i}(1-u)\int_0^u \breve{c}_{X_i/X_{(d-h)}}(\alpha,u,...,u)d\alpha = \beta(u).VaR_{X_i}(1-u) \tag{62}$$

Remark 1. *One can find a relation between the marginal function of dependence tail mean and marginal of the multivariate VaR proposed by Cousin[7], by using the previous corollary and proposition 2.4 in Cousin[7].*

Corollary 5. *Let X be a random vector, with multivariate distribution F and associeted copula C. Then, for all $u \in (0,1)$, $i = 1,...,h$*

i) the marginal function of expected dependence of lower tail verify,

$$\zeta^L_{h,i}(u) \leq (1-u)TVaR_{X_i}(u),$$

ii) the marginal function of expected dependence of upper tail verify,

$$\zeta^U_{h,i}(u) \leq (1-u)TVaR_{X_i}(1-u), \tag{63}$$

Proof.

i) We have,

$$\zeta^L_h(u) \quad = \quad \int_u^1 VaR_{X_i}(\alpha)c_{X_i/X_{(d-h)}}(\alpha,u,...,u)d\alpha \tag{64}$$

by using Hölder inequality we obtain,

$$\zeta^L_h(u) \leq \left(\int_u^1 VaR_{X_i}(\alpha)d\alpha\right)\|c_{X_i/X_{(d-h)}}(\alpha,u,...,u)\|_\infty.$$

In other hand, we have

$$TVaR_{X_i}(u) = \frac{1}{1-u}\int_u^1 VaR_{X_i}(\alpha)d\alpha. \tag{65}$$

So, we obtain

$$\begin{aligned}\zeta^L_h(u) &\leq (1-u)TVaR_{X_i}(u).\|c_{X_i/X_{(d-h)}}(\alpha,u,...,u)\|_\infty\\ &\leq (1-u)TVaR_{X_i}(u).\end{aligned} \tag{66}$$

since $0 \leq \|c_{X_i/X_{(d-h)}}(\alpha,u,...,u)\|_\infty \leq 1$, with $\|f\|_\infty = Sup_x\{\int|f(x)|dx\}$.

ii) we have,

$$\zeta_{h,i}^{U}(u) = \int_{0}^{1-u} VaR_{X_i}(1-\alpha).\breve{c}_{X_i/X_{(d-h)}}(\alpha, u, ..., u)d\alpha, \tag{67}$$

by using Hölder inequality we obtain.

$$\zeta_{h,i}^{U}(u) \leq \left(\int_{0}^{1-u} VaR_{X_i}(1-\alpha)d\alpha \right) \|\breve{c}_{X_i/X_{(d-h)}}(\alpha, u, ..., u)\|_{\infty},$$

in one other hand, we have

$$TVaR_{X_i}(1-u) = \frac{1}{1-u} \int_{0}^{1-u} VaR_{X_i}(1-\alpha)d\alpha, \tag{68}$$

we obtain

$$\begin{aligned} \zeta_{h}^{U}(u) &\leq (1-u)TVaR_{X_i}(1-u).\|\breve{c}_{X_i/X_{(d-h)}}(\alpha, u, ..., u)\|_{\infty} \\ &\leq (1-u)TVaR_{X_i}(1-u). \end{aligned} \tag{69}$$

since $0 \leq \|\breve{c}_{X_i/X_{(d-h)}}(\alpha, u, ..., u)\|_{\infty} \leq 1$

7. Conclusion and Discussion

The diversity of financial products and the interconnections between financial markets make investments increasingly risky. To avoid extreme losses or at least reduce their magnitude, it is necessary to acquire tools to properly model it. Through this present work, we have provided a contribution on this theme. In particular, the results offer us the possibility of modeling and describing the joint extreme behavior of several stochastic risks simultaneously.However, the applicability of some results, such as those on extremal processes, may seem problematic given their rare use in practice. There are works making point process applications and records [‡] in finance for example. Some researchers, as Resnick [19], give the way to construct extremal process based on Poisson point process, which can be helpful for application.

Moreover, extremal processes give us a time frame to model extreme events. The interesting fact is that they check the property of max-stability, thus making them strongly related to max-stable processes. Fortunately, the latter are widely used in stochastic modeling in various fields (hydrology, meteorology, geography, finance, etc.). Consequently, there are several models (temporal, spatial, spatio-temporal) of these processes with interesting results of applications. The particular advantage of extremal processes over max-stable ones is that they have distributions that can be expressed as a function of the time parameter. Therefore, for example, to make a spatio-temporal study of an extreme

[‡]"For the study of the stochastic behavior of maxima and records, extremal processes are a useful tool." Resnick [19] (section 4.3 P.179)

phenomenon with these processes, it suffices to integrate a spatial parameter.

The second important property of extremal processes is that of Markov. This property offers us the possibility of making predictions about the future values of the study variable. For example, This would make it possible to forecast the VaR, in particular the multivariate case, in the purely extreme setting (alternative to the mixed EVT-GARCH methods, etc.).

The Markov processes and the max-stable processes being popularized modeling objects, various estimation and inference methods exist in the literature and those in multitudes of fields including finance. Based on the aspects common to these types of processes, it would then be easy to adapt the extremal processes to applications. Extremal processes are better suited "naturally" to model the dynamics of extreme events as shown by the results of applications of simple EVT models ("deterministic") compared to non-extreme methods (Gaussian for example). It would be judicious to explore all of these possibilities through further practical studies to compare these approaches.

We have also defined two multivariate risk measures whose similar versions exist in the bivariate framework. Thus, by using of these two new measures, we can now measure the average occurrence of certain risks compared to others in the tails of the distributions. As example, if we consider a vector of two random variables $Z = (X, Y)$, modelizing the profits/losses of two assets, with Frchet margin and with dependance structure described by Gumbel-Hougaard's copula then the profits/losses average of one asset in the tails (lower and upper tails) is very important when the profits/losses of the second asset is superior or inferior of the Value at Risk (corrolary2).

Note also that the limits ζ_h^L and ζ_h^U can diverge since in Corollary 13, the functions $\zeta_h^L(u)$ and $\zeta_h^U(u)$ are greater than quantities which are expressed as the product of a finite quantity and the VaR. The latter can take infinite limit values and thus make these lower bounds quite wide. In perspective, it would be interesting to build consistent estimators for these risk measures in the context of extreme values for possible applications to real data.

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