New connections on the fiber-bundle of generalized statistical manifolds

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Abstract. In this paper we construct a family of α-connections on a fiber-bundle of a generalized statistical manifold. We prove that the exponential and mixture connections are curvature-free and we investigate the associated parallel transport.

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1 Introduction

Information geometry investigates the differential-geometric structure of statistical models and has many applications in statistical inference or machine learning for example (see [4]). Since the seminal work of Rao[14] where Fisher information is viewed as a Riemannian metric on a probability distributions space, statistical manifolds have been widely studied. The Fisher information metric on statistical manifold is related to the Kullback-Leibler divergence which is a measure of dissimilarity between two probability distributions. Considering a family of α-divergences, Amari[2] proposed a family of α-connections on statistical manifolds. To elucidate the structures and properties of estimating functions, Amari and Kumon[3] constructed a family of α-connections on Hilbert bundle of statistical manifold, endowed with the Fisher information metric. Vigelis et al.[17] introduced a new metric and α-connections using u_0-mappings (or φ-functions), which generalize Fisher information metric and Amari's α-connections. The obtained geometric structure is called a generalized statistical manifold. Recently, de Andrade et al.[5] investigated the mixture and exponential arcs on generalized statistical manifold.

In this paper, we extend the results of Vigelis et al.[17] by defining a new family of α-connections on Hilbert bundle of generalized statistical manifold. We prove that the curvatures of our proposed (1)-connection and (-1)-connection vanish everywhere. Moreover, we give the α-parallel transport associated with α-connection for $\alpha = \pm 1$.

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The rest of the paper is organized as follows. In section 2, we review the relevant concepts related to generalized statistical manifold. In section 3 we introduce our new α-connection on Hilbert bundle of generalized statistical manifold and prove the mains results.

2 Generalized statistical manifolds and α-connections

In this section, we recall some useful definitions and properties related to generalized statistic manifolds (see[17, 10, 16]). Let (χ, Σ, μ) be a measure space and $P_\mu = \{p \in L^0 : p > 0, \int_\chi p(x;\theta)d\mu(x) = 1\}$ where L^0 denotes the set of all real-valued, measurable functions on χ.

Definition 2.1. (see[17]) Let $u_0 : \chi \to (0, \infty)$ be a measurable function. A function $\phi : \mathbb{R} \to (0, \infty)$ is said to be a u_0-mapping if :

- ϕ is convex,
- $\lim_{x\to-\infty} \phi(x) = 0$ and $\lim_{x\to\infty} \phi(x) = \infty$,
- for all measurable function $c : \chi \to \mathbb{R}$ satisfying $\int_\chi \phi(c(x))d\mu(x) = 1$, we have $\int_\chi \phi(c(x) + \lambda u_0(x))d\mu(x) < \infty$, for all $\lambda > 0$.

Example 2.2. The function ϕ defined by

$$\phi(x) = \exp(ax + b), a \in (0; \infty), b \in \mathbb{R}, \quad \forall x \in \mathbb{R}$$

is a 1_χ-mapping.

Example 2.3. (see[16]) The Kaniadakis'κ-exponential $exp_\kappa : \mathbb{R} \to (0, \infty)$ defined by

- for $\kappa = 0$, $\exp_\kappa$ is the usual exponential map,
- for $\kappa \in [-1, 0[\cup]0, 1]$, $\exp_\kappa(x) = (\kappa x + \sqrt{1 + \kappa^2 x^2})^{1/\kappa}$

is a u_0-mapping where u_0 satisfies $\int_\chi exp_\kappa(u_0)d\mu < +\infty$.

Definition 2.4. (see [17]) Let ϕ be a smooth u_0-mapping. A generalized statistical manifold is a family of probability distributions

$$M = \{p(\cdot;\theta) : \theta \in \Theta\} \subset P_\mu$$

such that:

1. Θ is an open and connected set in $\mathbb{R}^n$.
2. Each $p(., \theta)$ is given in terms of $\theta \in \Theta$ by a one to one mapping.
3. Every function $p(x; \cdot)$ is smooth for all x and the operations of integration with respect to μ and differentiation with respect to θ^i (i.e. $\partial/\partial\theta^i$) are always commutative.
4. The support of $p(\cdot, \theta)$ does not depend on θ for all $\theta \in \Theta$.

5. The matrix $g = (g_{ij})$, which is defined by

$$g_{ij} = -\mathbb{E}'_\theta \left[\frac{\partial^2 f_\theta}{\partial \theta^i \partial \theta^j} \right]$$

is positive definite at each $\theta \in \Theta$, where $f_\theta(\cdot) = \phi^{-1}(p(\cdot;\theta))$ and

$$\mathbb{E}'_\theta [\cdot] = \frac{\int_\chi (\cdot)\phi^{(1)}(f_\theta)d\mu}{\int_\chi u_0 \phi^{(1)}(f_\theta)d\mu}. \tag{2.1}$$

g_{ij} is invariant under reparametrization. When ϕ is the usual exponential function and $u_0 = 1$, g is the Fisher information matrix.

Lemma 2.1. *[17] For $i, j \in \{1, 2, \cdots\}$ and $\theta \in M$,*

$$\mathbb{E}'_\theta \left[\frac{\partial f_\theta}{\partial \theta^i} \right] = 0 \quad \textit{and} \quad g_{ij} = \mathbb{E}''_\theta \left[\frac{\partial f_\theta}{\partial \theta^i} \frac{\partial f_\theta}{\partial \theta^j} \right],$$

where

$$\mathbb{E}''_\theta [\cdot] = \frac{\int_\chi (\cdot)\phi^{(2)}(f_\theta)d\mu}{\int_\chi u_0 \phi^{(1)}(f_\theta)d\mu}. \tag{2.2}$$

Let $\partial_i = \partial/\partial\theta^i$ be the tangent vector of the i-th coordinate curve θ^i at the point θ. Then, n such tangent vectors ∂_i, $i = 1, \cdots, n$, span the tangent space $\top_\theta M$ at the point θ of the manifold M. Any tangent vector $A \in \top_\theta M$ is a linear combination of the basis vectors ∂_i,

$$A = A^i \partial_i,$$

where A^i are the components of vector A and Einstein's summation convention is assumed throughout the paper. The tangent space $\top_\theta M$ is a linearized version of a small neighborhood of θ in M and an infinitesimal vector $d\theta = d\theta^i \partial_i$ denotes the vector connecting two neighboring points θ and $\theta + d\theta$ or two neighboring distributions $p(\cdot, \theta)$ and $p(\cdot, \theta + d\theta)$. Let us introduce a metric in the tangent space $\top_\theta M$. It can be done by defining the inner product of two basis vectors ∂_i and ∂_j. Usually the vector $\partial_i \in \top_\theta M$ is represented by a function $\partial_i f_\theta$ and the metric is defined by

$$g_{ij}(\theta) = g(\partial_i, \partial_j) = \langle \partial_i f_\theta, \partial_j f_\theta \rangle = \mathbb{E}''_\theta [\partial_i f_\theta \partial_j f_\theta], \forall \theta \in M \tag{2.3}$$

The α-covariant derivative (see [17]) of the basis vector ∂_j in the direction ∂_i is

$$\langle \nabla^{(\alpha)}_{\partial_i} \partial_j, \partial_k \rangle = \Gamma^{(\alpha)}_{ij,k}, \tag{2.4}$$

where

$$\Gamma^{(\alpha)}_{ij,k} = \frac{1+\alpha}{2} \Gamma^{(1)}_{ij,k} + \frac{1-\alpha}{2} \Gamma^{(-1)}_{ij,k},$$

$$\Gamma^{(1)}_{ij,k} = \mathbb{E}''_\theta \left[\frac{\partial^2 f_\theta}{\partial \theta^i \partial \theta^j} \frac{\partial f_\theta}{\partial \theta^k} \right] - \mathbb{E}'_\theta \left[\frac{\partial^2 f_\theta}{\partial \theta^i \partial \theta^j} \right] \mathbb{E}''_\theta \left[u_0 \frac{\partial f_\theta}{\partial \theta^k} \right],$$

$$\begin{aligned}\Gamma_{ij,k}^{(-1)} &= \mathbb{E}_\theta''\left[\frac{\partial^2 f_\theta}{\partial\theta^i\partial\theta^j}\frac{\partial f_\theta}{\partial\theta^k}\right]+\mathbb{E}_\theta'''\left[\frac{\partial f_\theta}{\partial\theta^i}\frac{\partial f_\theta}{\partial\theta^j}\frac{\partial f_\theta}{\partial\theta^k}\right]\\ &\quad -\mathbb{E}_\theta''\left[\frac{\partial f_\theta}{\partial\theta^j}\frac{\partial f_\theta}{\partial\theta^k}\right]\mathbb{E}_\theta''\left[u_0\frac{\partial f_\theta}{\partial\theta^i}\right]-\mathbb{E}_\theta''\left[\frac{\partial f_\theta}{\partial\theta^i}\frac{\partial f_\theta}{\partial\theta^k}\right]\mathbb{E}_\theta''\left[u_0\frac{\partial f_\theta}{\partial\theta^j}\right]\end{aligned}$$

and

$$\mathbb{E}_\theta'''[\cdot]=\frac{\int_\chi(\cdot)\phi^{(3)}(f_\theta)d\mu}{\int_\chi u_0\phi^{(1)}(f_\theta)d\mu}.$$

3 The Hilbert bundle of a generalized statistical manifold

3.1 Hilbert bundle

Let Υ be the set of μ-integrable and smooth functions r defined from χ to $\mathbb{R}$. Let ϕ be a smooth and bijective u_0-mapping $\phi:\mathbb{R}\to(0,\infty)$ such that $\forall(\theta,x)\in M\times(0,\infty)$, $\phi^{(1)}(x)\neq 0$ and $M=\{p(\ \cdot\ ;\theta);\theta=(\theta^1,\cdots,\theta^n)\in\Theta\subseteq\mathbb{R}^n\}$ a generalized statistical manifold endowed with the Riemannian metric g defined by (2.3) and parametrized by $\theta=(\theta^1,\cdots,\theta^n)$. To each point $\theta\in M$, we associate a linear space H_θ defined by

$$H_\theta=\{r\in\Upsilon:\mathbb{E}_\theta'[r]=0,\mathbb{E}_\theta''[r^2]<+\infty\},$$

where $\mathbb{E}_\theta'$ and $\mathbb{E}_\theta''$ are respectively defined by relations (2.1) and (2.2). Throughout this paper we assume that $\mathbb{E}_\theta''\left[(\partial_i f_\theta)^2\right]<\infty$ and for all $(\theta,\theta')\in M^2$, $r\in H_\theta$

$$E_\theta''[r^2]<+\infty\Longrightarrow\mathbb{E}_\theta''\left[r^2\left(\frac{\phi'(f_\theta)}{\phi'(f_{\theta'})}\right)^2\right]<+\infty.$$

For each $\theta\in M$ and $r,s\in H_\theta$ we set

$$\langle r,s\rangle_\theta:=\mathbb{E}_\theta''[rs]. \tag{3.1}$$

Proposition 3.1. *For all $\theta\in M$, $\langle\cdot,\cdot\rangle_\theta$ is an inner product and $(H_\theta,\langle\cdot,\cdot\rangle_\theta)$ is a Hilbert space.*

Proof. Let $\theta\in M$. H_θ is a vector space and the map $\langle\cdot,\cdot\rangle_\theta$ defined by (3.1) is a positive definite bilinear form, then it is a inner product on H_θ. Using the Walter's proof (see [18]) of completeness of the set of measurable and square integrable functions, one proves the completeness of H_θ. □

Since the tangent vectors $\partial_i f_\theta(x)$, which span $\top_\theta M$, satisfy $\mathbb{E}_\theta'[\partial_i f_\theta]=0$ and $\mathbb{E}_\theta''[(\partial_i f_\theta)^2]<+\infty$, they belong to H_θ. Indeed, the tangent space $\top_\theta M$ of M at θ is a linear subspace of H_θ, and the inner product defined in $\top_\theta M$ is compatible with that in H_θ. Let N_θ be the orthogonal complement of $\top_\theta M$ in H_θ. Then, H_θ is decomposed into the direct sum

$$H_\theta=\top_\theta M\oplus N_\theta.$$

The aggregate of all H_θ's attached to every $\theta \in M$ with a suitable topology,

$$H = \cup_{\theta \in M} H_\theta,$$

is called the fiber-bundle with base space M. Since the fiber space is a Hilbert space, it is called a Hilbert bundle of M. It should be noted that H_θ and $H_{\theta'}$ are different Hilbert spaces when $\theta \neq \theta'$.

Hence it is convenient to establish a one-to-one correspondence between H_θ and $H_{\theta'}$, when θ and θ' are neighboring points in M. When the correspondence is affine, it is called an affine connection. Let us assume that a vector $r \in H_\theta$ at θ corresponds to $r + dr \in H_{\theta+d\theta}$ at a neighboring points $\theta + d\theta$, where d denotes infinitesimally small change.

Lemma 3.2. *Let $\theta \in M$ and $r \in H_\theta$. Then*

$$\mathbb{E}'_\theta[dr] = -\mathbb{E}''_\theta[\partial_i f_\theta r] d\theta^i + o\left(\|d\theta\|\right).$$

Proof. Let $\theta \in M$ and $r \in H_\theta$. Then $r + dr \in H_{\theta+d\theta}$. Set $\Phi_x(\theta) = \phi^{(1)}(f_\theta(x))$, $x \in \chi$. The function Φ_x is differentiable on Θ. Then by Taylor expansion of the function $\theta \mapsto \Phi_x(\theta + d\theta)$ we obtain

$$\Phi_x(\theta + d\theta) = \Phi_x(\theta) + d_\theta \Phi_x(d\theta) + o(\|d\theta\|)$$

where $d_\theta \Phi_x$ denotes the differential of Φ_x at θ. Thus

$$\begin{aligned}
r + dr \in H_{\theta+d\theta} \quad &\Rightarrow \quad \mathbb{E}'_{\theta+d\theta}[r + dr] = 0 \\
&\Rightarrow \quad \int_\chi [r(x) + dr(x)] \Phi_x(\theta + d\theta) d\mu(x) = 0 \\
&\Rightarrow \quad \int_\chi [r(x) + dr(x)] \left(\Phi_x(\theta) + d_\theta \Phi_x(d\theta) + o(\|d\theta\|)\right) d\mu(x) = 0 \\
&\Rightarrow \quad \mathbb{E}'_\theta[dr] + \mathbb{E}''_\theta[r \partial_i f_\theta] d\theta^i \\
&\qquad + o(\|d\theta\|) \int_\chi [r(x) + dr(x)] d\mu(x) = 0. \qquad (3.2)
\end{aligned}$$

Using $\int_\chi |r(x) + dr(x)| d\mu(x) < \infty$ and (3.2) we deduce $\mathbb{E}'_\theta[dr] = -\mathbb{E}''_\theta[\partial_i f_\theta r] d\theta^i + o\left(\|d\theta\|\right)$. □

Next, we construct the α-connections on Hilbert bundle.

3.2 α-connections on Hilbert bundles

Given $r \in H_\theta$, differentiating the identity $\mathbb{E}'_\theta[r] = 0$ with respect to θ, we have $\mathbb{E}'_\theta[\partial_i r] = -\mathbb{E}''_\theta[\partial_i f_\theta r]$, $\mathbb{E}'[r] = \mathbb{E}'_\theta[\partial_i f_\theta] = 0$. Set

$$dr = \frac{1+\alpha}{2} \mathbb{E}'_\theta[\partial_i r] u_0 d\theta^i - \frac{1-\alpha}{2} \left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \partial_i f_\theta r - \mathbb{E}''_\theta[u_0 \partial_i f_\theta] r\right] d\theta^i.$$

We have

$$\begin{aligned}
\mathbb{E}'_\theta[dr] &= \mathbb{E}'\left[\frac{1+\alpha}{2}\mathbb{E}'_\theta[\partial_i r]u_0 d\theta^i - \frac{1-\alpha}{2}\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_i f_\theta r d\theta^i\right] \\
&= \frac{1+\alpha}{2}\mathbb{E}'_\theta[\partial_i r]d\theta^i - \frac{1-\alpha}{2}\mathbb{E}'\left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_i f_\theta r d\theta^i\right] \\
&= \frac{1+\alpha}{2}\mathbb{E}'_\theta[\partial_i r]d\theta^i - \frac{1-\alpha}{2}\mathbb{E}''\left[\partial_i f_\theta r\right]d\theta^i \qquad (3.3)\\
&= -\mathbb{E}''\left[\partial_i f_\theta r\right]d\theta^i.
\end{aligned}$$

The α-connection is given by the following α-covariant derivative $\bar{\nabla}^{(\alpha)}$. Let r be a vector field, which attaches a vector $r(\cdot,\theta)$ to every point $\theta \in M$. Then, the rate of the intrinsic change of the vector $r(\cdot,\theta)$ as θ changes in the direction ∂_i is given by the α-covariant derivative:

$$\bar{\nabla}^{(\alpha)}_{\partial_i} r = \partial_i r - \frac{1+\alpha}{2}\mathbb{E}'_\theta[\partial_i r]u_0 + \frac{1-\alpha}{2}\left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_i f_\theta r - \mathbb{E}''_\theta[u_0\partial_i f_\theta]r\right]. \qquad (3.4)$$

This is a generalization of the α-connection studied by Amari[1]. We have $\mathbb{E}'_\theta[\bar{\nabla}^{(\alpha)}_{\partial_i} r] = 0$ because of the identity $\partial_i\mathbb{E}'_\theta[r] = \mathbb{E}'_\theta[\partial_i r] + \mathbb{E}''_\theta[\partial_i f_\theta r]$. The α-covariant derivative in the direction $A = A^i\partial_i \in \top_\theta M$ is given by

$$\bar{\nabla}^{(\alpha)}_A r = A^i \bar{\nabla}^{(\alpha)}_{\partial_i} r.$$

The 1-connection is called the exponential connection and the -1-connection is called the mixture connection.

For each point $\theta \in M$, the tangent space $\top_\theta M$ is a subset of the Hilbert space H_θ. Hence the tangent bundle of M,

$$\top M = \cup_{\theta\in M}\top_\theta M,$$

is a subset of H. We can define an affine connection in $\top M$ by introducing an affine correspondence between $\top_\theta M$ and $\top_{\theta'} M$ for neighboring points θ and θ'.

An affine connection given such that $r \in H_\theta$ corresponds to $r + dr \in H_{\theta+d\theta}$, induces an affine connection in $\top_\theta M$ such that $r \in \top_\theta M \subset H_\theta$ corresponds to the orthogonal projection of $r + dr \in H_{\theta+d\theta}$ onto $\top_{\theta+d\theta}M$.

3.3 Parallel transport on Hilbert bundles

We start this subsection by the following definition.

Definition 3.1. Let $c = \{c(t), t \in [0,1]\}$ be a curve in M. A vector field $r(\cdot,t) \in H_{c(t)}$ defined along the curve c is said to be α-parallel, when

$$\bar{\nabla}^{(\alpha)}_{\dot{c}} r = \dot{r} - \frac{1+\alpha}{2}\mathbb{E}'_c[\dot{r}]u_0 + \frac{1-\alpha}{2}\left[\frac{\phi^{(2)}(f_c)}{\phi^{(1)}(f_c)}\dot{f}_c r - \mathbb{E}''_c[u_0\dot{f}_c]r\right] = 0, \qquad (3.5)$$

where $\dot{r}$ denotes $\partial r/\partial t$, etc.

Definition 3.2. A vector $r_1(\cdot) \in H_{\theta_1}$ is the α-parallel transport of $r_0(\cdot) \in H_{\theta_0}$ along a curve $c = \{c(t), t \in [0,1]\}$ connecting $\theta_0 = c(0)$ and $\theta_1 = c(1)$, when $r_0(\cdot) = r(\cdot, 0)$ and $r_1(\cdot) = r(\cdot, 1)$ in the solution $r(\cdot,\cdot)$ of (3.5).

Generally, the parallel transport along a curve c connecting θ to θ' depends on on c.

Theorem 3.3. *(see [12]) For an affine connection, parallel transport is independent of the path if and only if the curvature tensor vanishes.*

Now, we investigate the e- and m-parallel transport operators from H_θ to $H_{\theta'}$, for $(\theta, \theta') \in M^2$. Then we can prove the following important theorem.

Theorem 3.4. *Let ${}^{(e)}\pi_\theta^{\theta'}$ and ${}^{(m)}\pi_\theta^{\theta'}$ be the e- and m-parallel transport operators from H_θ to $H_{\theta'}$. Then*

$$\begin{aligned} {}^{(e)}\pi_\theta^{\theta'} r(x) &= r(x) - \mathbb{E}'_{\theta'}[r] u_0(x), \\ {}^{(m)}\pi_\theta^{\theta'} r(x) &= \frac{r(x)\phi^{(1)}(f_\theta(x)) \int_\chi u_0 \phi^{(1)}(f_{\theta'}) d\mu}{\phi^{(1)}(f_{\theta'}(x)) \int_\chi u_0 \phi^{(1)}(f_\theta) d\mu}. \end{aligned}$$

Proof. Let $c = \{c(t), t \in [0,1]\}$ be a curve connecting two points $\theta = c(0)$ and $\theta' = c(1)$. Let $r^{(\alpha)}(x,t)$ be an α-parallel transport vector defined along the curve c. Then, it satisfies (3.5). When $\alpha = 1$, it is reduced to

$$\frac{\dot{r}^{(e)}(x,t)}{u_0(x)} = \mathbb{E}'_{c(t)}\left[\dot{r}^{(e)}(\cdot,t)\right]. \tag{3.6}$$

Since the right-hand side

$$\mathbb{E}'_{c(t)}\left[\dot{r}^{(e)}(\cdot,t)\right] = \frac{\int_\chi \dot{r}^{(e)}(x,t)\phi^{(1)}\left(f_{c(t)}(x)\right) d\mu(x)}{\int_\chi u_0(x)\phi^{(1)}\left(f_{c(t)}(x)\right) d\mu(x)}$$

of (3.6) does not depend on x, its solution (with the initial condition $r(x) = r^{(e)}(x,0)$) is given by

$$r^{(e)}(x,t) = r(x) + a(t) u_0(x)$$

where

$$a(t) = -\mathbb{E}'_{c(t)}[r].$$

Then

$${}^{(e)}\pi_\theta^{\theta'} r(x) = r(x) - \mathbb{E}'_{\theta'}[r] u_0(x).$$

When $\alpha = -1$, (3.5) is reduced to

$$\dot{r}^{(m)}(x,t) + \left\{ \frac{\phi^{(2)}(f_{c(t)}(x))}{\phi^{(1)}(f_{c(t)}(x))} \dot{f}_{c(t)}(x) - \mathbb{E}''_{c(t)}[u_0 \dot{f}_{c(t)}] \right\} r^{(m)}(x,t) = 0.$$

The solution is

$$r^{(m)}(x,t) = k(x) \frac{\int_\chi u_0 \phi^{(1)}(f_{c(t)}(x)) d\mu}{\phi^{(1)}(f_{c(t)}(x))}$$

where $k(x) = \frac{r(x)\phi^{(1)}(f_\theta(x))}{\int_\chi u_0\phi^{(1)}(f_\theta(x))d\mu(x)}$. Then

$$^{(m)}\pi_\theta^{\theta'} r(x) = \frac{r(x)\phi^{(1)}(f_\theta(x))\int_\chi u_0\phi^{(1)}(f_{\theta'})d\mu}{\phi^{(1)}(f_{\theta'}(x))\int_\chi u_0\phi^{(1)}(f_\theta)d\mu}.$$

□

As consequence of the two previous theorems, we get the following result.

Corollary 3.5. *The exponential $\bar{\nabla}^{(1)}$ and mixture connection $\bar{\nabla}^{(-1)}$ are curvature free.*

Lemma 3.6. *We assume that $\forall x \in \chi$, $\phi^{(2)}(x) > 0$. If there exists a real constant k such that*

$$\phi^{(2)}(f_\theta) = \phi^{(1)}(f_\theta)\int_\chi u_0\phi^{(1)}(f_\theta)\,d\mu + k, \tag{3.7}$$

then

$$E_\theta'''\left[\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta\right] = \mathbb{E}_\theta''\left[\partial_i f_\theta \partial_k f_\theta\left(\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_j f_\theta + \mathbb{E}_\theta''(u_0\partial_j f_\theta)\right)\right]. \tag{3.8}$$

Proof. Let ϕ be a smooth and bijective u_0-mapping such that $\forall(\theta, x) \in M \times (0, \infty)$, $\phi^{(1)}(x) \neq 0$, $\phi^{(2)}(x) > 0$ and ϕ satisfies (3.7). Let $(i, j, k) \in \{1, 2, \cdots, n\}^3$.

$$\begin{aligned}
(3.7) \quad \Rightarrow \quad & \partial_j f_\theta \phi^{(3)}(f_\theta) = \partial_j f_\theta \phi^{(2)}(f_\theta)\int_\chi u_0\phi^{(1)}(f_\theta)\,d\mu \\
& +\phi^{(1)}(f_\theta)\int_\chi u_0\partial_j f_\theta\phi^{(2)}(f_\theta)\,d\mu \\
\Rightarrow \quad & \frac{\partial_j f_\theta\phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)} = \partial_j f_\theta\int_\chi u_0\phi^{(1)}(f_\theta)\,d\mu \\
& +\frac{\phi^{(1)}(f_\theta)}{\phi^{(2)}(f_\theta)}\int_\chi u_0\partial_j f_\theta\phi^{(2)}(f_\theta)\,d\mu \\
\Rightarrow \quad & \frac{\partial_j f_\theta\phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)} = \frac{\partial_j f_\theta\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \\
& +\frac{\phi^{(1)}(f_\theta)}{\phi^{(2)}(f_\theta)}\int_\chi u_0\partial_j f_\theta\phi^{(2)}(f_\theta)\,d\mu \\
\Rightarrow \quad & \partial_i f_\theta\partial_k f_\theta\frac{\partial_j f_\theta\phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)} = \partial_i f_\theta\partial_k f_\theta\left[\frac{\partial_j f_\theta\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\right. \\
(3.9) \qquad & \left.+\frac{\phi^{(1)}(f_\theta)}{\phi^{(2)}(f_\theta)}\int_\chi u_0\partial_j f_\theta\phi^{(2)}(f_\theta)\,d\mu\right].
\end{aligned}$$

By taking the expectation of (3.9), we get the relation (3.8). □

In the following theorem, we show that with a sufficient and necessary condition on ϕ, the restriction to $\top M$ of the family of α-connections (3.4) of H is exactly the family of α-connections (2.4) of $\top M$ given by Rui et al.[17]. Since the geometry of M is indeed that of $\top M$, our α-geometry of H is an extension of that of M.

Theorem 3.7. *For all i, j and k in $\{1, 2, \cdots, n\}$ and $\theta \in M$,*

$$g\left(\bar{\nabla}^{(\alpha)}_{\partial_i}\partial_j, \partial_k\right) = \Gamma^{(\alpha)}_{ij,k},$$

if and only if ϕ satisfies

$$E'''_\theta\left[\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta\right] = \mathbb{E}''_\theta\left[\partial_i f_\theta \partial_k f_\theta \left(\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_j f_\theta + \mathbb{E}''_\theta(u_0 \partial_j f_\theta)\right)\right],$$

where $\Gamma^{(\alpha)}_{ij,k}$ is defined by relation (2.4).

Proof. Let i, j and k in $\{1, 2, \cdots, n\}$. We have

$$\begin{aligned}
g\left(\bar{\nabla}^{(\alpha)}_{\partial_i}\partial_j, \partial_k\right) &= \left\langle \bar{\nabla}^{(\alpha)}_{\partial_i f_\theta}\partial_j f_\theta, \partial_k f_\theta\right\rangle_\theta \\
&= \left\langle \partial_i\partial_j f_\theta, \partial_k f_\theta\right\rangle_\theta - \frac{1+\alpha}{2}\mathbb{E}'_\theta[\partial_i f_\theta \partial_j f_\theta]\left\langle u_0, \partial_k f_\theta\right\rangle_\theta \\
&\quad + \frac{1-\alpha}{2}\left\langle \frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_i f_\theta \partial_j f_\theta, \partial_k f_\theta\right\rangle_\theta \\
&\quad - \frac{1-\alpha}{2}\left[\mathbb{E}''_\theta[u_0 \partial_i f_\theta]\left\langle \partial_j f_\theta, \partial_k f_\theta\right\rangle_\theta\right] \\
&= \mathbb{E}''_\theta\left[\partial_i\partial_j f_\theta \partial_k f_\theta\right] - \frac{1+\alpha}{2}\mathbb{E}'_\theta[\partial_i f_\theta \partial_j f_\theta]\mathbb{E}''_\theta\left[u_0 \partial_k f_\theta\right] \\
&\quad + \frac{1-\alpha}{2}\mathbb{E}''_\theta\left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta\right] \\
&\quad - \frac{1-\alpha}{2}\left\{\mathbb{E}''_\theta[u_0 \partial_i f_\theta]\mathbb{E}''_\theta\left[\partial_j f_\theta \partial_k f_\theta\right]\right\}.
\end{aligned}$$

We know that $\mathbb{E}''_\theta\left[\partial_i\partial_j f_\theta \partial_k f_\theta\right] = \frac{1+\alpha}{2}\mathbb{E}''_\theta\left[\partial_i\partial_j f_\theta \partial_k f_\theta\right] + \frac{1-\alpha}{2}\mathbb{E}''_\theta\left[\partial_i\partial_j f_\theta \partial_k f_\theta\right]$,

$$\begin{aligned}
\mathbb{E}'''_\theta\left[\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta\right] &= \mathbb{E}''_\theta\left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)}\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta\right] \\
&\quad + \mathbb{E}''_\theta\left[\partial_i f_\theta \partial_k f_\theta\right]\mathbb{E}''_\theta\left[u_0 \partial_j f_\theta\right].
\end{aligned}$$

Then one has $g\left(\bar{\nabla}^{(\alpha)}_{\partial_i}\partial_j, \partial_k\right) = \frac{1+\alpha}{2}\Gamma^{(1)}_{ij,k} + \frac{1-\alpha}{2}\Gamma^{(-1)}_{ij,k} = \Gamma^{(\alpha)}_{ij,k}$. □

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