4-index theory of gravity and its relation with the violation of the energy-momentum conservation law

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Recently, a 4-index generalization of the Einstein theory is proposed by Moulin [1]. Using this method, we find the most general 2-index field equations derivable from the Einstein-Hilbert action. The application of Newtonian limit, the role of gravitational coupling constant and the effects of the properties of ordinary energy-momentum tensor in obtaining a 4-index gravity theory have been studied. We also address the results of building Weyl free 4-index gravity theory. Our study displays that both the Einstein and Rastall theories can be obtained as the subclasses of a 4-index gravity theory which shows the power of 4-index method in unifying various gravitational theories. It is also obtained that the violation of the energy-momentum conservation law may be allowed in 4-index gravity theory, and moreover, the contraction of 4-index theory generally admits a non-minimal coupling between geometry and matter field in the Rastall way. This study also shows that, unlike the Einstein case, the gravitational coupling constant of 4-index Rastall theory generally differs from that of the ordinary 2-index Rastall theory.

I. INTRODUCTION

In Riemannian geometry, the geometrical information of a manifold is encoded into the forth-order Riemann tensor, while the general relativity (GR) includes the second rank divergence-less tensors [2, 3]. Remember that the Einstein tensor is a combination of the Ricci tensor and its scalar constructed by contracting the Riemann tensor.

Based on the Einstein hypothesis, whenever the spacetime is curved in the presence of an energy source, the amount of curvature is determinable by the total energy-momentum tensor of source filling the background. In fact, whenever the GR field equations are solved, then metric and thus the Riemann tensor are also determined, and we can find more information about spacetime by studying the Riemann tensor and its evolution. This approach is in fact a secondary way to study the Riemann tensor in which we do not know anything about the probable constraints on this tensor and its evolution. It means that GR does not directly say us anything about the evolution of this curvature. Thus, a proper theory should help us in obtaining the Riemann tensor and its evolution in a direct way and without any intermediary.

Theoretically, another primary way to directly find some information about the evolution of geometry may also be to establish a relation between the changes of geometry and that of the energy-momentum source filling it. In GR, at first glance, such a relation is exist between the derivatives of Ricci scalar (or Ricci tensor) and the trace of energy momentum tensor. But, this is a secondary relation directly obtained from the field equations which are the fundamental and basic equations in GR. Therefore, another basic assumptions are needed to build a theory which lets us directly study the evolution of geometry.

On the other hand, if one relaxes the energy-momentum conservation law by considering a mutual interaction between the geometry and matter field [4–11], then it can at least theoretically provide for us an independent basic to understand this issue that how the geometry is curved and its curvature evolves. Rastall theory is a leading work in this sense [4] which proposes a fundamental basic equation between the evolution of geometry and the changes of energy source, in a way it is ahead of its corresponding field equations. In fact, field equations in Rastall hypothesis, a modified GR, are secondary equations. Although this is an old theory, its Lagrangian is still under debate [9–13]. It is worthwhile mentioning that recent observation indicate that the gravitational waves are propagated with the light velocity [14], a result respected by both the Einstein and Rastall gravity [2, 15], and severely restricts gravitational theories as well as dark energy models [14]. Rastall gravity is inserted in the more general framework of extended gravity [19 - 24], which is today considered an intriguing tapestry to challenge the big puzzles of the standard model of cosmology, starting from the famous dark energy [25, 26] and dark matter [27, 28] problems. We emphasize that all of the potential alternatives to general relativity must be viable theories. This means that alternative theories must be consistent with Einstein's equivalence principle and, in turn, they must be metric theories [23]. In fact, Einstein's equivalence principle is today supported by an unchallengeable empirical

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evidence [23], being considered at the level of an important law of Nature. Another request is that alternative theories must pass the solar system tests. As a consequence, deviations of extended theories from standard general relativity must be weak [19, 20]. Remarkably, the nascent gravitational wave astronomy [29] could be a useful tool in order to discriminate among Einstein gravity, Rastall gravity and other potential alternative theories [15, 19, 20]. In fact, important differences between general relativity and extended theories can be pointed out in the linearized theory of gravity [19, 20].

Recently, a 4-index generalization of general relativity has been introduced to relate the Riemann tensor to the energy-momentum sources filling the spacetime [1]. Therefore, it is a theory may directly help us in modelling the evolution of geometry and thus its curvature. In this approach, the gravitational field contribution to the total energy-momentum tensor is separated from other sources, and it is related to the Weyl tensor defined as [3]

$$C_{ijkl}=R_{ijkl}-\frac{1}{n-1}(g_{ijkp}R^{p}{}_{l}+g_{ijpl}R^{p}{}_{k})+\frac{1}{n(n-1)}g_{ijkl}R, \tag{1}$$

and is zero for conformal flat spacetimes.

Therefore, the view of Ref [1] claims that the gravitational field has not any effects in conformal flat spacetimes which are indeed curved. If we accept the Einstein idea that spacetime is curved by energy sources, then this property of Weyl tensor will establishes an inconsistency with his idea. It is because conformal flat spacetimes are in fact curved whereas their Weyl tensor is zero. In fact, if physics is formulated in terms of this tensor, then some information will be disappeared whenever we face with conformal flat spacetimes such as the FRW geometry. It means that, in this situation, we should probably establish another set of equations to get the missed information meaning that the theory is incomplete. Briefly, this tensor does not has a unique behavior in front of the existence of curvature. Hence, although Weyl tensor includes some information about the geometry, due to its dual behavior against the existence of curvature, one may argue that a true 4-index generalization of Einstein theory should not include Weyl tensor.

Based on the above argument, we are going to show that the 4-index approach, introduced in [1], may provide a Lagrangian description for the Rastall theory. We are also interested in studying the role of gravitational coupling constant, the application of Newtonian limit and the results of building a Weyl tensor free gravitational field equations in this approach.

The paper is organized as follows. In the next section, after reviewing the lagrangian formalism in both 2 and 4-index notations, and addressing the role of gravitational coupling constant, we build the general 4-index gravitational field equations extractable by the Einstein-Hilbert Lagrangian. The conditions required for obtaining Einstein and Rastall theory have also been studied. Section (III) includes our surgery on the obtainable 2-index theories from this approach by generalizing the gravitational action used in the second section. The last section is devoted to a summary and concluding remarks.

II. ACTION, ITS VARIATION, 2-AND 4-INDEX THEORIES

Before focusing on our main aim, we review some features of the Lagrangian formalism of general relativity in $(n+1)$-dimension.

A. $(n+1)$-dimensional general relativity

The Einstein-Hilbert action is

$$I=I_G+I_m, \tag{2}$$

in which

$$I_G=-\frac{1}{2\kappa_n}\int R\sqrt{-g}\;d^{n+1}x, \tag{3}$$

is the gravitational action, I_m denotes the matter action [3], and

$$\kappa_n=\frac{2(n-1)\pi^{n/2}G_{n+1}}{(n-2)(\frac{n}{2}-1)!}, \tag{4}$$

is the $(n+1)$-dimensional Einstein coupling constant [17, 18]. Here,

$$G_{n+1}=2\pi^{1-n/2}\Gamma(\frac{n}{2})\frac{c^3\ell_p^{n-1}}{\hbar}, \tag{5}$$

is the $(n+1)$-dimensional Newtonian gravitational constant [16]. In this manner, applying the action principle to Eq. (2), the Einstein field equations are achieved as

$$G_{\mu\nu}=\kappa_n T_{\mu\nu}. \tag{6}$$

As a check, for $n=3$, we can easily find

$$\begin{aligned}&G_{3+1}=G_4=2\pi^{-1/2}\Gamma(\frac{3}{2})\frac{c^3\ell_p^2}{\hbar}=\frac{c^3\ell_p^2}{\hbar}\equiv G,\\&\kappa_3=8\pi G\equiv\kappa\\&I_G=-\frac{1}{2\kappa}\int R\sqrt{-g}\;d^4x,\end{aligned} \tag{7}$$

which finally leads to $G_{\mu\nu}=\kappa T_{\mu\nu}$.

The Newtonian limit

For a space with n dimension, the Poisson equation, corresponding to the Newtonian potential ϕ and energy density ρ, is written as [2, 17, 18]

$$\nabla^2\phi = \frac{2G_{n+1}\pi^{n/2}}{(\frac{n}{2}-1)!}\rho. \tag{8}$$

Moreover, the Newtonian limit is evaluated by using the [2]

$$R_{00} = \nabla^2\phi, \tag{9}$$

relation, for which Eq. (6) implies [2]

$$R_{00} = \left(\frac{n-2}{n-1}\right)\kappa_n\rho, \tag{10}$$

where in accordance with the properties of the Newtonian limit, the pressure contribution has been ignored [2]. Now, combining the above equations with each other, one can easily reach at Eq. (4). Thus, if a 4-index theory of gravity is available, then contracting the field equations and by using the resulting 2-index field equations, one can find R_{00} and thus the gravitational coupling constant of the primary 4-index theory.

B. 4-index notation, general remarks and the role of gravitational coupling constant

Since $R = g^{jl}g^{mn}R_{mjnl}$, by defining the 4-index metric g_{ijkl} as $g_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}$ which has the same symmetry as that of the Reimann tensor [1, 2], one obtains

$$\begin{aligned} R &= g^{ik}g^{jl}R_{ijkl} = \frac{1}{n}g^{ik}g^{jl}g_{ijkp}R^{p}{}_{l} \\ &= \frac{1}{n}g^{ik}g^{jl}g_{ijpl}R^{p}{}_{k} = \frac{1}{n(n+1)}g^{ik}g^{jl}g_{ijkl}R, \end{aligned} \tag{11}$$

which helps us in generalizing Eq. (3) as [1]

$$I_G = -\frac{1}{2\eta_n}\int g^{ik}g^{jl}(A+B+C+D)\sqrt{-g}\ d^{n+1}x, \tag{12}$$

where

$$\begin{aligned} A &= aR_{ijkl} \\ B &= mg_{ijkp}R^{p}{}_{l} \\ C &= mg_{ijpl}R^{p}{}_{k} \\ D &= dg_{ijkl}R, \end{aligned} \tag{13}$$

and a, m, and d are unknown Lagrangian coefficients evaluated later. In Ref [1], for both the $g_{ijkp}R^{p}{}_{l}$ and $g_{ijpl}R^{p}{}_{k}$ terms, author assumed the same coefficient (m). But, since $g_{ijkp}R^{p}{}_{l} \neq g_{ijpl}R^{p}{}_{k}$, their coefficients can be different in general. We will study some consequences of this case in the next section.

Now, using Eq. (11), one can see Eq. (3) is recovered if we have either

$$\begin{aligned} &n[2m+(n+1)d]+a = 1 \\ &\eta_n = \kappa_n, \end{aligned} \tag{14}$$

or

$$\eta_n \equiv \frac{\kappa_n}{\alpha}, \tag{15}$$

where $\alpha \equiv \frac{1}{n[2m+(n+1)d]+a}$. While the first case claims that the gravitational coupling constant in 4-index generalization of Einstein theory (η_n) is the same as κ_n, the second case indicates that η_n differs from κ_n. Therefore, the definition of gravitational coupling constant has a key role in getting a 4-index theory.

Finally, in similarity with the definition of Ricci tensor ($R_{jl} = g^{ik}R_{ijkl}$), and just the same as Ref [1], we assume that there is a 4-index generalized energy-momentum tensor T_{ijkl} satisfying the $T_{jl} = g^{ik}T_{ijkl}$ condition, in which T_{jl} is the ordinary 2-index energy-momentum tensor representing all sources filling the background and obtainable by applying the action principle to the matter Lagrangian, i.e. [1]

$$\begin{aligned} 2\delta I_m &= \int T_{jl}\delta g^{jl}\sqrt{-g}\ d^{n+1}x \\ &= \int T_{ijkl}g^{ik}\delta g^{jl}\sqrt{-g}\ d^{n+1}x. \end{aligned} \tag{16}$$

Action variation and 4-index theory

Following the method of Ref [1], the variation of action (12) leads to

$$\begin{aligned} \delta I_G = -\frac{1}{2\eta_n}\int g^{ik}\big[&(m(n-1)+a)\delta R_{ik} + \delta g^{jl}\big(\\ &aR_{ijkl} + m(g_{ijkp}R^{p}{}_{l} + g_{ijpl}R^{p}{}_{k}) \\ &+(d - \frac{a+2mn+dn(n+1)}{2n})g_{ijkl}R\big) \\ &+(m+nd)\delta(g_{ik}R)\big]\sqrt{-g}\ d^{n+1}x. \end{aligned} \tag{17}$$

The integral of $g^{ik}\delta R_{ik}$ will be vanished [1], and the coefficient of the $\delta(g_{ik}R)$ will be zero whenever $m = -dn$, the primary and simple case studied in [1]. In this manner, combining this result with Eq. (16), one reaches at

$$G_{ijkl} \equiv \big[aR_{ijkl} - nd(g_{ijkp}R^p_{\ l} + g_{ijpl}R^p_{\ k}) + (\frac{dn(1+n)-a}{2n})g_{ijkl}R\big] = \eta_n T_{ijkl}. \quad (18)$$

Now, using Eq. (1), we can rewrite this equation as

$$G_{ijkl} = \big[aC_{ijkl} + F_{ijkl}\big] = \eta_n T_{ijkl}, \quad F_{ijkl} = \frac{a-n(n-1)d}{n-1}(g_{ijkp}R^p_{\ l} + g_{ijpl}R^p_{\ k}) + \frac{n+1}{2n(n-1)}[dn(n-1)-a]g_{ijkl}R. \quad (19)$$

The Weyl free case

The above result indicates that the field equations will be free of Weyl tensor whenever $a = 0$ leading to

$$G_{ijkl} = F^{(a=0)}_{ijkl} = \eta_n T_{ijkl}. \quad (20)$$

In this manner, since we assumed that T_{jl} includes all sources filling the background, we do not need additional terms to cancel the Weyl tensor. Additionally, although F_{ijkl} is completely determinable by the Ricci tensor and metric, we cannot find the 4-index energy-momentum tensor unless we have relation between T_{jl} and Ricci tensor meaning that we should decide about the desired 2-index theory.

General Relativity

The contraction of Eq. (20) leads to

$$G_{jl} = \frac{\eta_n}{n(1-n)d}T_{jl}, \quad (21)$$

nothing but the Einstein field equations with the coupling constant $\frac{\eta_n}{n(1-n)d}$. Now, comparing this equation with (6), we can easily see that the results of (II A) is also valid here for $\frac{\eta_n}{n(1-n)d} = \kappa_n$, a result also compatible with Eq. (15). Therefore, this analysis cannot give us the values of η_n and d meaning that their values should be evaluated from other parts. In fact, this analysis shows that the Newtonian limit and the $d = -\frac{m}{n}$ constraint are enough to recover the Einstein field equations whenever the 4-index equations are Weyl free. As an example, if $\eta_n \equiv \kappa_n$, then we should have $n(n-1)d = -1$ in full agreement with Eq. (14) and Ref [1].

In order to find T_{ijkl} and η_n, we remind that 4-index energy-momentum tensor should meet the $T_{jl} = g^{ik}T_{ijkl}$ condition. One can use Eqs. (20) and (21) in order to see that only if $n(1-n)d = 1$ and

$$T_{ijkl} = \frac{1}{n-1}(g_{ijkp}T^p_{\ l} + g_{ijpl}T^p_{\ k}) - \frac{1}{n(n-1)}g_{ijkl}T, \quad (22)$$

then the $T_{jl} = g^{ik}T_{ijkl}$ condition is met in agreement with Ref [1]. In this situation, from the $\frac{\eta_n}{n(1-n)d} = \kappa_n$ relation, we automatically reach at $\eta_n = \kappa_n$. Moreover, inserting the above results into Eq. (20), we easily get [1]

$$B_{ijkl} \equiv F^{(a=0,n(1-n)d=1)}_{ijkl} = -\frac{n+1}{2n(n-1)}g_{ijkl}R + \frac{1}{n-1}(g_{ijkp}R^p_{\ l} + g_{ijpl}R^p_{\ k}) = \kappa_n T_{ijkl}. \quad (23)$$

which meets [1]

$$G_{jl} = g^{ik}B_{ijkl}. \quad (24)$$

We see that the $T_{jl} = g^{ik}T_{ijkl}$ condition together with the Newtonian limit automatically give us the value of d leading to $\eta_n = \kappa_n$. Now, since the energy-momentum conservation law is met by T_{jl} in Einstein theory, one can obtain

$$\nabla_i T^i_{\ jkl} = \frac{1}{n-1}\big[(T_{jl;k} - T_{jk;l}) - \frac{1}{n}(g_{jl}T_{,k} - g_{jk}T_{,l})\big]. \quad (25)$$

Bearing the Einstein field equations and Eq. (20) in mind and using the above result, one finds

$$\nabla_i T^i_{\ jkl} = \frac{1}{\kappa_n}\nabla_i G^i_{\ jkl} = \frac{1}{\kappa_n(n-1)}\big[(R_{jl;k} - R_{jk;l}) + \frac{1}{2n}(g_{jk}R_{,l} - g_{jl}R_{,k})\big], \quad (26)$$

which is not always zero [1]. Therefore, this study shows that while there is no non-minimal mutual interaction between the geometry and matter fields in 2-index general relativity (or equally $T^i_{\ j;i} = 0$), its 4-index generalization admits a non-minimal coupling between them meaning that the divergence of the 4-index energy-momentum tensor is not always zero.

C. Some notes on the $m \neq -nd$ case

Here, we want to address the results of the $m \neq -nd$ case. Let us focus on the last term of Eq. (17). In fact, whenever $m \neq -nd$, then since $g^{ik}g_{ik} = n+1$, we have

$$g^{ik}\delta(g_{ik}R) = \delta(g^{ik}g_{ik}R) - (\delta g^{ik})Rg_{ik} = (n+1)\delta R - Rg_{jl}\delta g^{jl}, \quad (27)$$

Now, because $\delta R \rightarrow G_{jl}\delta g^{jl} = g^{ik}B_{ijkl}\delta g^{jl}$, and $g_{jl} = \frac{1}{n}g_{ijkl}g^{ik}$, the last term of Eq. (27) leads to

$$g^{ik}\delta(g_{ik}R) = g^{ik}[(n+1)B_{ijkl} - R\frac{1}{n}g_{ijkl}]\delta g^{jl}, \quad (28)$$

apart of the usual surface term which is the result of the δR term and will be zero at infinity [1–3]. In this manner, bearing Eq. (1) in mind, one finally reaches at

$$\begin{aligned} &G_{ijkl} \equiv \left[aC_{ijkl} + \mathbb{F}_{ijkl}\right] + (m+nd)(n+1)B_{ijkl} \\ &= \eta_n T_{ijkl}, \quad (29) \\ &\mathbb{F}_{ijkl} = (m + \frac{a}{n-1})(g_{ijkp}R^p_{\ l} + g_{ijpl}R^p_{\ k}) \\ &-(\gamma + \frac{a}{n(n-1)})g_{ijkl}R. \\ &\gamma \equiv \frac{(dn+2m)(1+n)+a}{2n}. \end{aligned}$$

Now, contracting this equation ($g^{ik}G_{ijkl} = \eta_n g^{ik}T_{ijkl} = \eta_n T_{jl}$), we find out

$$\begin{aligned} &G_{jl} + \Xi_n\lambda R g_{jl} = \Xi_n T_{jl}, \\ &\lambda = \frac{m(n+1) - 2\gamma n + a}{2\eta_n}, \quad (30) \\ &\Xi_n = \frac{\eta_n}{a + n(m + nd(n+1))}, \end{aligned}$$

leading to $R(4\Xi_n\lambda - 1) = \Xi_n T$, the trace of field equations. The above field equations are indeed the Rastall field equations in which λ and Ξ_n denote the Rastall constant and Rastall gravitational coupling constant, respectively [4]. It is also worthwhile mentioning that the $T^i_{\ j;i} = 0$ condition is not met by the Rastall theory claiming that the geometry and matter fields are coupled with each other in a non-minimal way [4, 5]. It should be noted that since we face with Rastall field equations in which Ξ_n is the gravitational coupling constant, replacing κ_n with Ξ_n in Eq. (15), one can reach the above results. λ and Ξ_n are also connected to each other by considering the Newtonian limit of the Rastall theory [4]as $\frac{\Xi_n}{4\Xi_n\lambda - 1}(3\Xi_n\lambda - \frac{1}{2}) = \frac{\kappa_n}{2}$. This result can also be obtained by following the recipe introduced in (II A). It finally leads to $\eta_n = \frac{4m - n(8\gamma + d(n+1)) + 2a}{10m - n(12\gamma + d(n+1)) + 5a}\kappa_n$ which clearly indicates that we do not have always $\eta_n = \kappa_n$ [4]. The Newtonian limit indeed helps us in finding relation between Ξ_n with G.

It is also easy to check that for $m = -nd$, we get $\lambda = 0$ and thus the Einstein field equations are recovered (for which $\Xi_n = \kappa_n$), as a desired result in full agreement with previous achievements. Therefore, in this manner, the contraction of the 4-index field equations automatically presents a mutual interaction between geometry and matter field, i.e. $T_{\mu\nu}^{\ \ ;\nu} = \lambda R_{,\mu}$.

Field equations (29) will also become Weyl free whenever $a = 0$. In this manner, bearing the recipe led to Eq. (22) in mind, one can use Eq. (30) and the trace of field equations to find B_{ijkl} and $\mathbb{F}_{ijkl}$ which finally leads to T_{ijkl}.

III. A MORE GENERAL LAGRANGIAN

Here, after introducing a generalization to Eq. (13), we introduce another 4-index generalization for Rastall theory based on the $m = -nd$ case. Some results of the Weyl tensor free theory are also discussed.

A. Action and its variation

As we addressed previously, since $g_{ijkp}R^p_{\ l} \neq g_{ijpl}R^p_{\ k}$, their coefficients can be different in general. Here, considering different coefficients for these terms, we are going to study some consequences of the resulting action. Let us consider a more general form for action (12) by generalizing Eq. (13) as

$$\begin{aligned} &A = aR_{ijkl}, \quad (31) \\ &B = mg_{ijkp}R^p_{\ l}, \\ &C = cg_{ijpl}R^p_{\ k}, \\ &D = dg_{ijkl}R, \end{aligned}$$

where c (the same as a, m, and d) is an unknown Lagrangian coefficient. In this situation, Eqs. (14) and (15) are modified as

$$\begin{aligned} &n[m + c + (n+1)d] + a = 1 \\ &\eta_n = \kappa_n, \quad (32) \end{aligned}$$

and

$$\eta_n \equiv \frac{\kappa_n}{\beta}, \quad (33)$$

in which $\beta \equiv \frac{1}{n[m+c+(n+1)d]+a}$, respectively. The action variation also leads to

$$\begin{aligned} &\delta I_G = -\frac{1}{2\eta_n}\int g^{ik}\big[(cn - m + a)\delta R_{ik} + \delta g^{jl}\big(\\ &aR_{ijkl} + mg_{ijkp}R^p_{\ l} + cg_{ijpl}R^p_{\ k} \quad (34) \\ &+(d - \frac{1}{2n\beta})g_{ijkl}R\big) + (m+nd)\delta(g_{ik}R)\big]\sqrt{-g}\ d^{n+1}x, \end{aligned}$$

and thus

$$\begin{aligned} &G_{ijkl} \equiv \big[aR_{ijkl} - ndg_{ijkp}R^p_{\ l} + cg_{ijpl}R^p_{\ k} \\ &+(\frac{n(d-c) - a}{2n})g_{ijkl}R\big] \\ &= \eta_n T_{ijkl}, \quad (35) \end{aligned}$$

where, as the previous section, we considered the $m = -nd$ case for which the $\delta(g_{ik}R)$ term is eliminated. Now, bearing the Weyl tensor (1) in mind, we can finally reach

$$G_{ijkl} = \left[aC_{ijkl} + \mathcal{F}_{ijkl}\right] = \eta_n T_{ijkl}, \quad (36)$$
$$\mathcal{F}_{ijkl} = f_1 g_{ijkp} R^p_{\ l} + f_2 g_{ijpl} R^p_{\ k} + f_3 g_{ijkl} R.$$

Here,

$$f_1 \equiv \frac{a - n(n-1)d}{n-1},\ f_2 \equiv \frac{a + c(n-1)}{n-1},$$
$$f_3 \equiv \frac{n(d-c)(n-1) - a(n+1)}{2n(n-1)}, \quad (37)$$

and it is easy to see that Eq. (19) is recovered at the appropriate limit of $c = -nd$.

B. Rastall theory

In the general framework of extended gravity, which has been discussed in the Introduction of this paper, a renewed interest in the literature has been recently gained by the theory proposed by P. Rastall in 1972 [4]. In fact, Rastall theory of gravity presents various good behaviors. It seems consistent with the Universe age and with the Hubble parameter [30], with the helium nucleosynthesis [31] and with the gravitational lensing phenomena [32]. It permits an alternative description for the matter dominated era with respect to general relativity [33]. Such observational evidences enabled cosmologists to study the various cosmic eras in the framework of Rastall gravity [34 - 39]. In addition, Rastall gravity should not present the entropy and age problems of standard cosmology [40]. As we previously stressed, the fundamental issue concerning Rastall gravity is the presence of a non-divergence-free energy-momentum. For the sake of completeness, we recall that also the so called curvature-matter non-minimal theory of gravity shows a similar behavior because also in this theory the matter and geometry are coupled to each other in such a way that the ordinary-energy momentum conservation law is not met [5–7, 40–42].

Now, let us restart our discussion. In general, since $g^{ik}C_{ijkl} = 0$, by contracting Eq. (36), we can obtain

$$G_{jl} + \Xi_n \lambda R g_{jl} = \Xi_n T_{jl}, \quad (38)$$

as the most general obtainable 2-index field equations. As the previous section, Ξ_n and λ denote the Rastall constant and Rastall gravitational coupling constant, respectively, but here, they are evaluated as

$$\Xi_n \equiv \frac{\eta_n}{a - n^2 d - c}, \quad (39)$$
$$\lambda \equiv \frac{1-n}{2\eta_n}[c + nd].$$

Once again, we see that the contraction of 4-index theory guides us in general to a non-minimal interaction between geometry and matter fields in the Rastall way.

We can also easily see that the $c = m = -nd$ case (or equally $\lambda = 0$) leads to the Einstein field equations. Therefore, as an expected result, the Einstein case can be considered as the subclass ($c = m = -nd$) of the Lagrangian introduced here (31), and thus this section. The Newtonian limit of the Rastall theory implies $\frac{\Xi_n}{4\Xi_n\lambda - 1}(3\Xi_n\lambda - \frac{1}{2}) = \frac{\kappa_n}{2}$ [4], combined with Eq. (39) to obtain $\eta_n = \frac{(a-n^2d-c)[c(3-2n)+d(2-n)-a]}{c(4-3n)+dn(3-2n)-a}\kappa_n$ indicating that we do not have always $\eta_n = \kappa_n$. Therefore, even for $m = -nd$, the 4-index formalism, introduced in Ref [1], can provide a Lagrangian description for Rastall theory [9, 12, 13].

The Weyl free case ($a = 0$)

In this manner ($a = 0$), ane can finally find T_{ijkl} by using Eq. (38) and the definition of $\mathcal{F}_{ijkl}$ as

$$T_{ijkl} = -\frac{[f_1 g_{ijkp} T^p_{\ l} + f_2 g_{ijpl} T^p_{\ k} - f_4 g_{ijkl} T]}{n^2 d + c}, \quad (40)$$

where

$$f_4 \equiv \frac{1}{4\Xi_n\lambda - 1}\left(\frac{2c + n(d-c)}{-2(n^2d + c)}[f_1 + f_2]^{a=0} - f_3^{a=0}\right) \quad (41)$$

If we want to obtain the same T_{ijkl} as that of the Einstein case (22), then the Lagrangian coefficients are constrained by

$$f_1^{a=0} = f_2^{a=0} = nf_4 = -\frac{n^2 d + c}{n-1}, \quad (42)$$

leading to $c = -nd$. As we saw in previous section, the $m = c = -nd$ case can only cover the Einstein field equations. Therefore, although both the Rastall and Einstein theory has the same T_{jl} (16), and they can be classified as the subclasses of one Lagrangian with the same T_{jl} meeting Eq. (16), their 4-index energy-momentum tensor is different. A difference which is the result of the existence of a non-minimal mutual interaction between the geometry and matter fields in Rastall theory.

It is also worthwhile mentioning that the $T_{jl} = g^{ik}T_{ijkl}$ condition is satisfied whenever we have $c = nf_4$ leading to $c = \frac{dn}{n-2}$. Inserting this result into Eq. (39), one easily finds $\Xi_n = \frac{\eta_n(2-n)}{nd(n(n-2)+1)}$ and $\lambda = \frac{nd(n-1)^2}{2\eta_n(2-n)}$. This achievement indicates that the gravitational coupling constant of 4-index Rastall theory (η_n) differs from that of the ordinary 2-index Rastall theory (Ξ_n), a result in agreement with subsection (II C). Thus, these coupling constants will be the same only if we have $d = \frac{2-n}{n(n(n-2)+1)}$ leading to $c = \frac{1}{n(2-n)-1}$.

Just the same as the Einstein case, Eq. (40) indicates that we have not always $\nabla_i T^i{}_{jkl} = 0$, meaning that there is also a non-minimal coupling between the geometry and matter fields in 4-index generalization of Rastall theory, a result in accordance with the Rastall hypothesis [4]. In fact, since $T^i{}_{j;i} = \lambda R_{,j} = \frac{\Xi_n \lambda}{4\Xi_n \lambda - 1} T_{,j}$ in Rastall theory [4], by bearing the definition of $\mathcal{F}_{ijkl}$ in mind (36), and combining the energy-momentum conservation law of Rastall theory with Eqs. (36) and (40), one can easily check that

$$\nabla_i G^i{}_{jkl} = \eta_n \nabla_i T^i{}_{jkl}. \quad (43)$$

Indeed, since $a = 0$ leading to $\mathcal{F}_{ijkl} = \eta_n T_{ijkl}$, the validity of this result has been guaranteed. Therefore, just the same as the original Rastall theory [4], the energy-momentum conservation law is not always satisfied in the 4-index generalization of this theory.

IV. CONCLUSION

In summary, our survey shows that although the $d = -\frac{m}{n}$ constraint is sufficient to recover the field equations of general relativity (Eqs. (24) and (21)), the fulfilment of the $T_{jl} = g^{ik} T_{ijkl}$ requirement necessitates $\eta_n = \kappa_n$. It has been obtained that $\nabla_i T^i{}_{jkl} \neq 0$ in general, meaning that a non-minimal coupling between geometry and matter fields is automatically allowed in 4-index approach, a result independent of the divergence amount of energy-momentum tensor ($T_{jl}{}^{;l}$). In addition, we also found out that the contraction of the 4-index field equations may in general bring us to a non-minimum interaction between geometry and matter field. It has also been obtained that, unlike the Einstein case, the gravitational coupling constant of 4-index Rastall theory (η_n) generally differs from that of the ordinary 2-index Rastall theory (Ξ_n).

Besides, we addressed some general expressions for 4-index energy-momentum tensor, depending on the values of the Lagrangian coefficients, which can reduce to those of the Einstein and Rastall cases by choosing their related coefficients. Therefore, it should be noted that the 4-index energy-momentum tensor of Rastall differs from that of Einstein, while T_{jl} is evaluated from a unique action principle in both theory (see Eq. (16) and Ref [11]). This difference is in line with the difference between the 2-index Rastall and Einstein theories.

We also found out that Rastall field equations can be obtained from Lagrangian (12) if $m \neq -nd$, a result claiming that both the Rastall and Einstein theory are subclasses of one general Lagrangian. On the other hand, in the third section, generalizing the Lagrangian (12), we could again get a 4-index generalization for the Rastall theory and thus a Lagrangian description of this theory. Moreover, although in our calculations $m = -nd$, we saw that the results of section (II) can be obtained as the special case ($c = -nd$) of this section. This result also indicates that both the Einstein and Rastall theory can be considered as the subclasses of one 4-index theory.

It is worthwhile to remind that *i*) the Riemann tensor has a crucial role in the Riemannian geometry, and *ii*) the key point in writing the Lagrangian (12) is Eq. (11) giving us the some possibilities of writing R using the g^{ik} and g_{ijkp} metrics together with the Riemann tensor and its contracted form (the Ricci tensor). In the Riemannian geometry, there are also another curvature invariants built by the Reimann tensor such as the Kretschmann scalar [2] and the Carminati-McLenaghan invariants [43]. Hence, writing R (or equally, the Lagrangian of the geometrical part of the Einstein theory) in terms of other invariants, and by using the g_{ijkp} notion, one may get another Lagrangians instead of (12). In this manner, one may find another 4-index field equations which should cover the Einstein field equations after contraction, and also the Newtonian gravity after taking the weak field limit. It was not our aim to study such possibilities, and can be considered as an interesting subject for the future works.

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