Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 12 (2019), 611–620
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

On the attractivity of an integrodifferential system with state-dependent delay

Kora Hafiz Bete[a], Carlos Ogouyandjou[a], Amadou Diop[b], Mamadou Abdoul Diop[b,*]

[a]Université d'Abomey-Calavi, Institut de Mathématiques et de Sciences Physiques, 01 B.P. 613, Porto-Novo, Bénin.

[b]Université Gaston Berger de Saint-Louis, UFR SAT, Département de Mathématiques, B.P. 234, Saint-Louis, Sénégal.

Abstract

This work is focused on the existence and attractivity of mild solutions for an integrodifferential system with state-dependent delay. The results presented here were established by means of a fixed point theorem due to [T. A. Burton, C. Kirk, Math. Nachr., **189** (1998), 23–31]. At the end, the obtained results are illustrated by an example.

Keywords: Neutral functional integrodifferential equations, resolvent operator, mild solution, local attractivity, fixed point theory, infinite delay.

2010 MSC: 34G20, 34K10, 34K30.

1. Introduction

Many phenomena in real life as thermodynamics, electrokinetics, biology, and chemistry can not be described by the classical differential equations. For these phenomena, the integrodifferential equations are more appropriate.

These last years, integrodifferential problems have been of great interest to several researchers. Thus, several works have been done by many researchers on integrodifferential equations (see for example, [1, 7–10], and references therein).

On the other hand, finite delays and their effects, play a very important role in differential equations. In many fields they are needed in the description of complex physical, biological or chemical phenomena (one can see [14, 19] for details). Moreover, to describe (integro)differential delayed systems, Hale et al. introduced the deterministic functional delayed differential systems, which are many theoretical and practical interests (see [13]).

However, in many areas of science, there is a growing interest in studying infinitely delayed integrodifferential equations. The study of such equations strongly depends on the choice of a phase space.

*Corresponding author
Email addresses: betekora.hafiz@imsp-uac.org (Kora Hafiz Bete), ogouyandjou@imsp-uac.org (Carlos Ogouyandjou), diop.amadou@ugb.edu.sn (Amadou Diop), mamadou-abdoul.diop@ugb.edu.sn (Mamadou Abdoul Diop)

doi: 10.22436/jnsa.012.09.06

Received: 2018-11-18 *Revised:* 2019-02-22 *Accepted:* 2019-04-04

Indeed, several phase spaces exist in the literature and each of these spaces has its specificities (see [16]). But the most used is the one developed by Hale and Kato (see [12]).

The study of existence, uniqueness, stability, controllability, and attractiveness is one of the most interesting topics of the qualitative theory of (integro)differential equations.

Attractiveness is a very important notion of the theory of dynamical systems, introduced for the first time by the Russian mathematician and engineer, Lyapunov in his thesis (see [18]). But since its introduction in 1892, this field has been greatly developed. Recently, many results on existence and attractiveness of mild solutions for several types of differential problems in infinite dimensional spaces, using various approaches were obtained (for more details, see [2–4]).

However, the literature about local attractivity results for neutral integrodifferential system with state-dependent delay is not large.

Motivated by the aforementioned problems, in this work, using a fixed method, we consider the existence of mild solution and attractivity for the following delayed, neutral integrodifferential system in a Banach space

$$\begin{cases} \frac{d}{dt}\left[x(t)-k(t,x_{\rho(t,x_t)})\right] \\ \quad = A[x(t)-k(t,x_{\rho(t,x_t)})]+\int_0^t \Upsilon(t-s)\left[x(s)-k(s,x_{\rho(s,x_s)})\right]ds+h(t,x_{\rho(t,x_t)}), & t\in J:=[0,\infty), \\ x(t)=\phi(t) & t\in(-\infty,0], \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ generates a C_0-semigroup $\{T(t),\ t \geqslant 0\}$ on a Banach space $(X, \|\cdot\|)$, $(\Upsilon(t))_{t\geqslant 0}$ is a family of closed operators on X, having a domain $D(\Upsilon) \supset D(A)$ which is independent of t, $x_t \in \mathcal{B}$ is a function defined from $(-\infty, 0]$ into X by $x_t(\theta) = x(t+\theta)$ for $\theta \in (-\infty, 0]$, where $\mathcal{B}$ denotes the abstract phase space, described axiomatically, h and k are given functions from $J \times \mathcal{B}$ into X, $\rho : J \times \mathcal{B} \to \mathbb{R}$ is an appropriate function, $\phi \in \mathcal{B}$.

The paper has four sections. In Section 2, we give some basic definitions, lemmas and the general theory of integrodifferential equations. Section 3 deals with the existence of the mild solution under certain conditions. In Section 4, we discuss about the attractivity of the solution obtained in the previous section. In Section 5, we give an example to illustrate the obtained results.

2. Preliminaries and the general theory of partial integrodifferential equations in Banach spaces

Throughout this paper, X is a Banach space with norm $\|\cdot\|$. A and Υ are closed linear operators defined on X. Let us denote by $BC(J, X)$ the space of all the maps x which are continuous and bounded from J into X. When we endow $BC(J, X)$ with the supremum norm $\|x\|_{BC} = \sup\limits_{t\in J} \|x(t)\|$, then $BC(J, X)$ is a Banach space. This norm will also be denoted by $\|\cdot\|$ if there is no possible confusion.

Let χ be the space:

$$\chi = \{x : \mathbb{R} \to X \text{ such that } x|_J \in BC(J, X) \text{ and } x_0 \in \mathcal{B}\}.$$

We denote by $x|_J$ the restriction of x to J. We introduce the Banach space $Y = (D(A), \|\cdot\|_Y)$, where $\|\cdot\|_Y$ denotes the graph norm defined by $\|y\|_Y = \|Ay\| + \|y\|$ for $y \in Y$.

We denote by $\mathcal{C}(\mathbb{R}_+, Y)$, the space of all functions from $\mathbb{R}_+$ into Y which are continuous.

Now, we are interested in the problem of Cauchy below:

$$\begin{cases} L'(t) = AL(t) + \int_0^t \Upsilon(t-s)L(s)ds \text{ for } t \geqslant 0, \\ L(0) = l_0 \in X. \end{cases} \tag{2.1}$$

The solution $\{L(t),\ t \geqslant 0\}$ satisfying the above differential system is an X-valued process.

Definition 2.1 ([11]). A bounded linear operator valued function $Q(t) \in \mathcal{L}(X)$ for $t \geqslant 0$, is said to be the resolvent operator of (2.1) if it satisfies the following conditions:

1. $Q(0) = I$ and $\|Q(t)\|_{\mathcal{L}(X)} \leqslant Me^{\beta t}$ for some constants M and β;
2. $\forall\, x \in X$, $Q(t)x$ is strongly continuous for $t \in \mathbb{R}_+$;
3. $Q(t) \in \mathcal{L}(Y)$ for $t \geqslant 0$ and $Q(\cdot)x \in \mathcal{C}^1(\mathbb{R}_+, X) \cap \mathcal{C}(\mathbb{R}_+, Y)$, for $x \in Y$.

$$Q'(t)x = AQ(t)x + \int_0^t \Upsilon(t-s)Q(s)x\,ds,$$
$$Q'(t)x = Q(t)Ax + \int_0^t Q(t-s)\Upsilon(s)x\,ds, \quad t \geqslant 0.$$

Next, we introduce the following hypotheses:

(C1) The operator A generates a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on X.

(C2) For all $t \geqslant 0$, the operator $\Upsilon(t)$ is closed and linear from $D(A)$ to X and $\Upsilon(t) \in L(Y, X)$. For any $y \in X$, the map $t \mapsto \Upsilon(t)y$ is bounded, differentiable and the derivative $t \mapsto \Upsilon(t)y$ is bounded and uniformly continuous for $t \geqslant 0$. In addition, there is a function $\mu : J \to \mathbb{R}_+$ which is integrable such that for each $z \in X$, the map $t \to \Upsilon(t)z$ belongs to $W^{1,1}(J, X)$ and $\left\|\frac{d\Upsilon(t)z}{dt}\right\| \leqslant \mu(t)\|z\|$, $z \in X$, $t \in J$.

Theorem 2.2 ([11])**.** *Let* (C1)-(C2) *hold. Then the Cauchy system* (2.1) *possesses a resolvent operator.*

We give the following important estimate.

Lemma 2.3 ([17])**.** *Let* (C1) *and* (C2) *be satisfied. Then there is a constant* L *such that*

$$\|Q(t+\epsilon) - Q(\epsilon)Q(t)\|_{\mathcal{L}(X)} \leqslant L\epsilon.$$

In what follows, we give an axiomatic definition of the phase space $\mathcal{B}$ due to Hale and Kato (see [12]) and we use the same terminology as in [16].

Definition 2.4. We denote by $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ the seminormed linear space of functions defined on $(-\infty, 0]$ into X, which satisfy the following axioms.

(A1) If $x : (-\infty, b) \to X$ is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then for each $t \in [0, b)$ the following conditions are satisfied:
- **(i)** $x_t \in \mathcal{B}$;
- **(ii)** $\|x(t)\| \leqslant H\|x_t\|_{\mathcal{B}}$ where $H > 0$ is a constant;
- **(iii)** $\|x_t\|_{\mathcal{B}} \leqslant \gamma(t)\sup\{\|x(s)\| : 0 \leqslant s \leqslant t\} + \lambda(t)\|x_0\|_{\mathcal{B}}$, where $\gamma(\cdot), \lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ are independent of x with γ continuous and λ locally bounded.

(A2) Let x be the function considered in (A1), x_t is a $\mathcal{B}$-valued function which is continuous on $[0, b]$.

(A3) The space $\mathcal{B}$ is complete.

Remark 2.5. Concerning the functions γ and λ, we suppose that they are bounded on J and

$$\kappa := \max\left\{\sup_{t\in\mathbb{R}_+}\{\gamma(t)\},\ \sup_{t\in\mathbb{R}_+}\{\lambda(t)\}\right\}.$$

Lemma 2.6 ([6])**.** *Let us consider the subspace* $\mathcal{C}$ *of* $BC(J, X)$ *satisfying*

(i) $\mathcal{C}$ *is bounded in* $BC(J, X)$;
(ii) *the functions which belong to* $\mathcal{C}$ *are equicontinuous on any compact of* J;
(iii) *the set* $\mathcal{C}(t) := \{x(t) : x \in \mathcal{C}\}$ *is relatively compact on any compact of* J;
(iv) *the functions from* $\mathcal{C}$ *are equiconvergent, this is, for* $\epsilon > 0$, *there is* $\Sigma(\epsilon) > 0$ *such that* $\|x(t) - x(+\infty)\| < \epsilon$ *for any* $t \geqslant \Sigma(\epsilon)$ *and* $x \in \mathcal{C}$.

Then $\mathcal{C}$ *is relatively compact in* $BC(J, X)$.

We give the Burton-Kirk's fixed-point theorem, which is very important in the proof of our results.

Theorem 2.7 ([5]). *In the Banach space* X, *let us consider the operators* Ψ_1 *and* Ψ_2 *defined from* X *into* X *such that* Ψ_1 *is a compact operator and* Ψ_2 *a contraction. Then either*

(i) $x = \delta\Psi_2\left(\frac{x}{\delta}\right) + \delta\Psi_1 x$ *admits a solution for* $\delta = 1$; *or*
(ii) *the set* $\{x \in X : x = \delta\Psi_2\left(\frac{x}{\delta}\right) + \delta\Psi_1 x,\ \delta \in (0,1)\}$ *is unbounded.*

Throughout this work, we suppose that (C1) and (C2) hold.

3. Existence of mild solutions

Definition 3.1. A function $x \in \chi$ is called mild solution of system (1.1) if x satisfies:

1. $x(t) = \phi(t)$, if $t \leqslant 0$;
2. $x(t) = Q(t)\left(\phi(0) - k(0,\phi)\right) + k(t, x_{\rho(t,x_t)}) + \int_0^t Q(t-s)h(s, x_{\rho(s,x_s)})ds$ for $t \geqslant 0$.

In addition, we make the following assumptions in order to get the expected results.

(H1) $(Q(t))_{t\geqslant 0}$ is compact and there is a constant $M \geqslant 1$ and $\gamma > 0$ satisfying

$$\|Q(t)\|_{\mathcal{L}(X)} \leqslant Me^{-\gamma t} \text{ for every } t \geqslant 0.$$

(H2) For a.e. $t \in J$ and each $u \in \mathcal{B}$, there is a function $f \in L^1(J, \mathbb{R}_+)$ satisfying:

$$\|h(t,u)\| \leqslant f(t)(\|u\|_{\mathcal{B}} + 1).$$

(H3) For each $t \geqslant 0$,

$$\lim_{t\to\infty} \int_0^t e^{-\gamma t} f(s)ds = 0.$$

(H4) For all $t,\ s \in J$ and $\phi,\ \varphi \in \mathcal{B}$, there is a constant $l > 0$ such that

$$\|k(t,\phi) - k(s,\varphi)\| \leqslant l(|t-s| + \|\phi - \varphi\|_{\mathcal{B}}).$$

(H5) $\forall\ t \in J,\ \phi \in \mathcal{B}$, $\|k(t,\phi)\| \leqslant \pi(t)\|\phi\|_{\mathcal{B}}$, where $\pi : J \to \mathbb{R}_+$ is a bounded continuous function.
(H6) Let $\mathcal{R}_\rho^- := \{\rho(t,x_t) : \rho(t,x_t) \leqslant 0,\ (t,x_t) \in \mathcal{B}\}$ and $\phi_t : \mathcal{R}_\rho^- \to \mathcal{B}$ a continuous function. There exists a bounded continuous map $L^\phi : \mathcal{R}_\rho^- \longrightarrow (0,\infty)$ such that $\|\phi_t\| \leqslant L^\phi(t)\|\phi\|_{\mathcal{B}}$ for all $t \in \mathcal{R}_\rho^-$.

Remark 3.2. For more detailed information concerning (H6) see Lemma 3.4 and Proposition 7.1.1 in [16].

The following lemma is needed to prove our results.

Lemma 3.3 ([15, Lemma 3.1]). *Let* $x : (-\infty,\infty) \to X$ *be continuous and bounded map and* $x_0 = \phi$. *If* (H6) *holds, then*

$$\|x_t\|_{\mathcal{B}} \leqslant (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{\|x(\theta)\|;\ \theta \in [0, \max\{0,t\}]\},\ t \in \mathcal{R}_\rho^- \cup J,$$

with $L^\phi = \sup\{L^\phi(t),\ t \in \mathcal{R}_\rho^-\}$.

Theorem 3.4. *Assume that assumptions* (C1)-(C2), (H1)-(H6) *hold. Also assume that*

$$\kappa l < 1 \quad \text{and} \quad \kappa\left(\sup_{t\in J}\|\pi(t)\| + M\right) < 1.$$

Then system (1.1) *has at least a mild solution.*

Proof. We introduce the following operator $\Sigma : \chi \to \chi$ defined by:

$$\Sigma x(t) = \begin{cases} \phi(t), & \text{if } t \leqslant 0, \\ Q(t)\,(\phi(0) - k(0,\phi)) + k(t, x_{\rho(t,x_t)}) + \int_0^t Q(t-s)h(s, x_{\rho(s,x_s)})ds, & \text{if } t \in J. \end{cases}$$

We show that Σ has a fixed point. For $\phi \in \mathcal{B}$, let $\alpha : \mathbb{R} \to X$ be the function given by

$$\alpha(t) = \begin{cases} \phi(t), & \text{if } t \leqslant 0, \\ Q(t)\,(\phi(0)), & \text{if } t \in J. \end{cases}$$

Then $\alpha_0 = \phi$. For each function $\vartheta \in \chi$, we denote

$$x(t) = \alpha(t) + \vartheta(t).$$

This implies that $x_t = \alpha_t + \vartheta_t$ and the function $\vartheta(\cdot)$ satisfies

$$\vartheta(t) = Q(t)k(0,\phi) + k(t, \alpha_{\rho(s,\alpha_s+\vartheta_s)} + \vartheta_{\rho(s,\alpha_s+\vartheta_s)}) + \int_0^t Q(t-s)h(s, \alpha_{\rho(s,\alpha_s+\vartheta_s)} + \vartheta_{\rho(s,\alpha_s+\vartheta_s)})ds.$$

In the following, we set χ_0 as Banach space defined by

$$\chi_0 = \{\vartheta \in \chi : \vartheta_0 = 0\},$$

with the norm

$$\|\vartheta\|_{\chi_0} = \sup_{t\in J} \|\vartheta(t)\| + \|\vartheta_0\|_{\mathcal{B}} = \sup_{t\in J} \|\vartheta(t)\|.$$

We introduce the decomposition $\vartheta(t) = \Psi_1\vartheta(t) + \Psi_2\vartheta(t)$, where

$$\begin{aligned} Psi_1\vartheta(t) &= \int_0^t Q(t-s)h(s, \alpha_{\rho(s,\alpha_s+\vartheta_s)} + \vartheta_{\rho(s,\alpha_s+\vartheta_s)})ds, \text{ for } t \in J, \\ \Psi_2\vartheta(t) &= Q(t)k(0,\phi) + k(t, \alpha_{\rho(s,\alpha_s+\vartheta_s)} + \vartheta_{\rho(s,\alpha_s+\vartheta_s)}). \end{aligned}$$

Showing the existence of solution of system (1.1) is equivalent to prove the existence of a fixed point of $\Psi_1 + \Psi_2$. Before this, we give the following estimate.

For each $\vartheta \in \chi_0$ and $t \in J \cup \mathcal{R}_\rho^-$, we have

$$\begin{aligned} \|\vartheta_t + \alpha_t\|_{\mathcal{B}} \leqslant \|\vartheta_t\|_{\mathcal{B}} + \|\alpha_t\|_{\mathcal{B}} &\leqslant \gamma(t)\|\vartheta(t)\| + \gamma(t)\|Q(t)\|\|\phi\|_{\mathcal{B}} + (\lambda(t) + L^{\phi})\|\phi\|_{\mathcal{B}} \\ &\leqslant \kappa\|\vartheta\|_{\chi_0} + \kappa M e^{-\gamma t}\|\phi\|_{\mathcal{B}} + (\kappa + L^{\phi})\|\phi\|_{\mathcal{B}} \\ &\leqslant \kappa\|\vartheta\|_{\chi_0} + [\kappa(M+1) + L^{\phi}]\|\phi\|_{\mathcal{B}}. \end{aligned} \tag{3.1}$$

First, we show the continuity of Ψ_1. Let $(\vartheta^{\tilde{k}})_{k\in\mathbb{N}}$ be a sequence in χ_0 such that $\vartheta^{\tilde{k}} \to \vartheta$ in χ_0, then for every $t \in J$ we have

$$\begin{aligned} &\|\Psi_1(\vartheta^{\tilde{k}})(t) - \Psi_1(\vartheta)(t)\| \\ &\leqslant \int_0^t \|Q(t-s)\|_{\mathcal{L}(X)} \left\| h(t, \alpha_{\rho(t,\alpha_s+\vartheta_s^{\tilde{k}})} + \vartheta^{\tilde{k}}_{\rho(t,\alpha_s+\vartheta_s^{\tilde{k}})}) - h(t, \alpha_{\rho(s,\alpha_s+\vartheta_s)} + \vartheta_{\rho(s,\alpha_s+\vartheta_s)}) \right\| ds \\ &\leqslant M \int_0^t e^{-\gamma(t-s)} \left\| h(s, \vartheta^{\tilde{k}}_{\rho(t,\alpha_s+\vartheta_s^{\tilde{k}})} + \alpha_{\rho(t,\alpha_s+\vartheta_s^{\tilde{k}})}) - h(s, \vartheta_{\rho(t,\alpha_s+\vartheta_s)} + \alpha_{\rho(t,\alpha_s+\vartheta_s)}) \right\| ds. \end{aligned}$$

By using the assumptions on the function h and the Lebesgue dominated convergence theorem we get

$$\|\Psi_1\vartheta^{\tilde{k}} - \Psi_1\vartheta\|_{\chi_0} \xrightarrow[k\to\infty]{} 0,$$

which shows that Ψ_1 is continuous.

Second, let $\eta \geqslant 0$ such that $\mathcal{S} = \{\vartheta \in \chi_0 : \|\vartheta\|_{\chi_0} \leqslant \eta\}$ is a bounded set. We prove that $\Psi_1(\mathcal{S})$ is relatively compact using Lemma 2.6. Let $\vartheta \in \mathcal{S}$, then

$$\begin{aligned}\|\Psi_1(\vartheta)(t)\| &\leqslant \int_0^t \|Q(t-s)\|_{\mathcal{L}(X)} \left\|h\left(s, \vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)}\right)\right\| ds \\ &\leqslant M \int_0^t e^{-\gamma(t-s)} f(s) \left(\|\vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}} + 1\right) ds \\ &\leqslant M \left(\kappa\|\vartheta\|_{\chi_0} + (\kappa(M+1) + L^{\phi})\|\phi\|_{\mathcal{B}} + 1\right) \int_0^t e^{-\gamma(t-s)} f(s) ds \leqslant M\xi\|f\|_{L^1},\end{aligned}$$

with

$$\xi := \kappa\eta + (\kappa(M+1) + L^{\phi})\|\phi\|_{\mathcal{B}} + 1.$$

Thus $\Psi_1(\mathcal{S})$ is bounded.

Third, we prove that $\Psi_1(\mathcal{S})$ is equicontinuous.

Let $s, t \in [0, b]$ with $t > s$ and $\vartheta \in \mathcal{S}$. Then, we have

$$\begin{aligned}\|(\Psi_1\vartheta)(t) - (\Psi_1\vartheta)(s)\| &= \bigg\| \int_0^s (Q(t-\tau) - Q(s-\tau)) h(\tau, \vartheta_{\rho(\tau,\alpha_\tau+\vartheta_\tau)} + \alpha_{\rho(\tau,\alpha_\tau+\vartheta_\tau)}) d\tau \\ &\quad + \int_s^t (Q(t-\tau)) h(\tau, \vartheta_{\rho(\tau,\alpha_\tau+\vartheta_\tau)} \quad + \alpha_{\rho(\tau,\alpha_\tau+\vartheta_\tau)}) d\tau \bigg\| \\ &\leqslant \int_0^s \|Q(t-\tau) - Q(s-\tau)\|_{\mathcal{L}(X)} f(\tau) \left(\|\vartheta_{\rho(\tau,\alpha_\tau+\vartheta_\tau)} + \alpha_{\rho(\tau,\alpha_\tau+\vartheta_\tau)}\|_{\mathcal{B}} + 1\right) d\tau \\ &\quad + M \int_s^t e^{-\gamma(t-\tau)} f(\tau) \left(\|\vartheta_{\rho(\tau,\alpha_\tau+\vartheta_\tau)} + \alpha_{\rho(\tau,\alpha_\tau+\vartheta_\tau)}\|_{\mathcal{B}} + 1\right) d\tau.\end{aligned}$$

Now, by the inequality (3.1) we get

$$\|(\Psi_1\vartheta)(t) - (\Psi_1\vartheta)(s)\| \leqslant \xi \int_0^s \|Q(t-\tau) - Q(s-\tau)\|_{\mathcal{L}(X)} f(\tau) d\tau + M\xi \int_s^t f(\tau) d\tau \xrightarrow[t-s\to 0]{} 0,$$

which implies the equicontinuity of $\Psi_1(\mathcal{S})$.

Now, we will prove the relative compactness of $\Gamma := \{(\Psi_1\vartheta)(t) : \vartheta \in \mathcal{S}\}$ in X. Let $t \in J$ be fixed and $\epsilon \in \mathbb{R}$ be a number such that $0 < \epsilon < t \leqslant b$. For $\vartheta \in \mathcal{S}$ we define

$$\Psi_1^{\star\epsilon}(\vartheta)(t) = Q(\epsilon) \int_0^{t-\epsilon} Q(t-\epsilon-s) h(s, \vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)}) ds$$

and

$$\tilde{\Psi}_1^{\star\epsilon}(\vartheta)(t) = \int_0^{t-\epsilon} Q(t-s) h(s, \vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)}) ds.$$

Using Lemma 2.3 and the compactness of the operator $Q(\epsilon)$, we deduce that the set $\Gamma_\epsilon^1(t) = \{\Psi_1^{\star\epsilon}(\vartheta)(t) : \vartheta \in \mathcal{S}\}$ is relatively compact in X. Moreover, also by Lemma 2.3 and Hlder inequality for each $\vartheta \in \mathcal{S}$, we obtain

$$\begin{aligned}&\|\Psi_1^{\star\epsilon}(\vartheta)(t) - \tilde{\Psi}_1^{\star\epsilon}(\vartheta)(t)\| \\ &\quad \leqslant \int_0^{t-\epsilon} \|Q(\epsilon)Q(t-\epsilon-s) - Q(t-s)\|_{\mathcal{L}(X)} \left\|h(s, \vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)})\right\| ds \\ &\quad \leqslant \int_0^{t-\epsilon} \|Q(\epsilon)Q(t-\epsilon-s) - Q(t-s)\|_{\mathcal{L}(X)} f(s) \left(\|\vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}} + 1\right) ds.\end{aligned}$$

So the set $\tilde{\Gamma}^1_\epsilon(t) = \{\tilde{\Psi}_1^{\star\epsilon}(\vartheta)(t) : \vartheta \in \mathcal{S}\}$ is precompact in X by using the total boundedness. Using this idea again, we obtain

$$\begin{aligned}\|\Psi_1(\vartheta)(t) - \tilde{\Psi}_1^{\star\epsilon}(\vartheta)(t)\| \leqslant & \int_{t-\epsilon}^{t} \|Q(t-s)\|_{\mathcal{L}(X)} \left\|h(s, \vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)})\right\| ds \\ \leqslant & \int_{t-\epsilon}^{t} \|Q(t-s)\|_{\mathcal{L}(X)} f(s) \left(\|\vartheta_{\rho(s,\alpha_s+\vartheta_s)} + \alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}} + 1\right) ds \\ \leqslant & \xi M \int_{t-\epsilon}^{t} e^{-\gamma(t-s)} f(s) ds \xrightarrow[\epsilon\to 0]{} 0.\end{aligned}$$

So, $\bar{\Psi}_1^{\star\epsilon}(\vartheta)$ converges uniformly to $\Psi_1(\vartheta)$ and it follows that $\mathcal{S}(t)$ is precompact in X.

Finally, it remains to show that Ψ_1 is equiconvergent.

Let $\vartheta \in \mathcal{S}$, using assumptions (H1), (H2), and (3.1) we obtain

$$\|(\Psi_1\vartheta)(t)\| \leqslant \xi M \int_0^t e^{-\gamma(t-s)} f(s) ds.$$

By (3.1) it follows that $\|(\Psi_1\vartheta)(t)\| \to 0$ as $t \to +\infty$. Then

$$\lim_{t\to\infty} \|(\Psi_1\vartheta)(t) - (\Psi_1\vartheta)(+\infty)\| = 0$$

and thus Ψ_1 is equiconvergent.

Finally, we show that Ψ_2 is a contraction.

Let $\vartheta, \tilde{\vartheta} \in \chi_0$, then by (H1), (H4)-(H6), and (3.1) and for each $t \in J$ we have

$$\begin{aligned}\|(\Psi_2\vartheta)(t) - (\Psi_2\tilde{\vartheta})(t)\| \leqslant & \left\|k(t, \alpha_{\rho(t,\alpha_t+\vartheta_t)} + \vartheta_{\rho(t,\alpha_t+\vartheta_t)}) - k(t, \alpha_{\rho(t,\alpha_t+\tilde{\vartheta}_t)} + \tilde{\vartheta}_{\rho(t,\alpha_t+\tilde{\vartheta}_t)})\right\| \\ \leqslant & l\|\vartheta_{\rho(t,\alpha_t+\vartheta_t)} - \tilde{\vartheta}_{\rho(t,\alpha_t+\tilde{\vartheta}_t)}\|_{\mathcal{B}}.\end{aligned}$$

So, by Lemma 3.3 we obtain

$$\|(\Psi_2\vartheta)(t) - (\Psi_2\tilde{\vartheta})(t)\| \leqslant \kappa l \|\vartheta - \tilde{\vartheta}\|_{\chi_0}.$$

Therefore

$$\|\Psi_2\vartheta - \Psi_2\tilde{\vartheta}\|_{\chi_0} \leqslant \kappa l \|\vartheta - \tilde{\vartheta}\|_{\chi_0}.$$

Thus, the operator Ψ_2 is a contraction.

To apply Theorem 2.7, we must verify that condition (ii) of Theorem 2.7 does not hold, i.e., prove that the set

$$\mathcal{S}_\delta = \left\{\vartheta \in \chi_0 : \vartheta = \delta\Psi_2\left(\frac{\vartheta}{\delta}\right) + \delta\Psi_1(\vartheta) \text{ for } \delta \in (0,1)\right\},$$

is bounded. Let $\vartheta \in \mathcal{S}_\delta$, then for each $t \in J$, we have

$$\vartheta(t) = \delta\Psi_2\left(\frac{\vartheta}{\delta}\right)(t) + \delta\Psi_1(\vartheta)(t).$$

Then we get

$$\begin{aligned}\|\vartheta(t)\| \leqslant & \delta\|Q(t)\|_{\mathcal{L}(X)}\|k(0,\phi)\| + \delta\left\|k(t, \alpha_{\rho(t,\alpha_t+\vartheta_t)} + \frac{\vartheta_{\rho(t,\alpha_t+\vartheta_t)}}{\delta})\right\| \\ & + \delta\int_0^t \|Q(t-s)\|_{\mathcal{L}(X)} \left\|h(s, \alpha_{\rho(s,\alpha_s+\vartheta_s)} + \vartheta_{\rho(s,\alpha_s+\vartheta_s)})\right\| ds \\ \leqslant & \delta M\pi(t)\|\phi\|_{\mathcal{B}} + \delta\pi(t)\left\|\alpha_{\rho(t,\alpha_t+\vartheta_t)} + \frac{\vartheta_{\rho(t,\alpha_t+\vartheta_t)}}{\delta}\right\|_{\mathcal{B}}\end{aligned}$$

$$+\delta M\int_0^t e^{-\gamma(t-s)}f(s)(\|\alpha_{\rho(s,\alpha_s+\vartheta_s)}+\vartheta_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}}+1)ds.$$

Thus, by Lemma 3.3 we have

$$\|\vartheta(t)\|\leqslant M\pi^*\|\phi\|_{\mathcal{B}}+\pi^*\kappa(\|\vartheta\|_{\chi_0}+\mu)+M\kappa(\|\vartheta\|_{\chi_0}+\mu+1)\|f\|_{L^1},$$

with $\mu:=(M+1)\|\phi\|_{\mathcal{B}}$ and $\pi^*:=\sup_{t\in J}\|\pi(t)\|$. Therefore

$$\|\vartheta\|_{\chi_0}\leqslant\frac{M\pi^*\|\phi\|_{\mathcal{B}}+\pi^*\kappa\mu+M(\mu+1)\|f\|_{L^1}}{(1-\pi^*\kappa-M\kappa)}:=c.$$

This shows that $\mathcal{S}_\delta$ is bounded. So, by Theorem 2.7 the operator Σ admits at least a fixed point, corresponding to a mild solution of the system (1.1) $\square$

4. Local attractivity of the solutions

This section concerns the local attractivity of the mild solutions of system (1.1). We have the following definition.

Definition 4.1. Mild solutions of system (1.1) are said to be locally attractive if there is a closed ball $\overline{B}(\vartheta^*,\Xi)$ in the space χ_0 for some $\vartheta^*\in\chi$ such that for arbitrary solutions ϑ and $\tilde{\vartheta}$ of (1.1) which belong to $\overline{B}(\vartheta^*,\Xi)$ we have

$$\lim_{t\to\infty}(\vartheta(t)-\tilde{\vartheta}(t))=0.$$

Let us denote by $\overline{B}(\vartheta^*,\Xi)$ the closed ball in χ_0, where ϑ^* is a solution of (1.1) and Ξ satisfies

$$\Xi\geqslant\frac{2M\|f\|_{L^1}}{1-l\kappa-2M\kappa\|f\|_{L^1}}.$$

Moreover, assume that

$$\lim_{t\to\infty}\pi(t)=0 \text{ and } \lim_{t\to\infty}\int_0^t e^{-\gamma(t-s)}\pi(s)ds=0,\tag{4.1}$$

where π is the function mentioned in (H6). Then, for $\vartheta\in\overline{B}(\vartheta^*,\Xi)$ by (H1)-(H2) and (3.1) we have

$$\begin{aligned}
&\|(\Sigma\vartheta)(t)-\vartheta^*(t)\|\\
&\quad=\|(\Sigma\vartheta)(t)-(\Sigma\vartheta^*)(t)\|\\
&\quad\leqslant\left\|k(t,\vartheta_{\rho(t,\alpha_t+\vartheta_t)}+\alpha_{\rho(t,\alpha_t+\vartheta_t)})-k(t,\vartheta^*_{\rho(t,\alpha_t+\vartheta_t)}+\alpha_{\rho(t,\alpha_t+\vartheta_t)})\right\|\\
&\qquad+\int_0^t\|Q(t-s)\|_{\mathcal{L}(X)}\left\|h(s,\vartheta_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)})-h(s,\vartheta^*_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)})\right\|ds\\
&\quad\leqslant l\|\vartheta_{\rho(t,\alpha_t+\vartheta_t)}-\vartheta^*_{\rho(t,\alpha_t+\vartheta_t)}\|_{\mathcal{B}}\\
&\qquad+M\int_0^t e^{-\gamma(t-s)}f(t)\left(\|\vartheta_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}}+\|\vartheta^*_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}}+2\right)ds\\
&\quad\leqslant l\kappa\Xi+2M(\kappa\Xi+1)\|f\|_{L^1}\leqslant\Xi.
\end{aligned}$$

Therefore, we get $\Sigma(\overline{B}(\vartheta^*,\Xi))\subset\overline{B}(\vartheta^*,\Xi)$. Thus, for each $\vartheta\in\overline{B}(\vartheta^*,\Xi)$ as the solution of system (1.1) and $t\in J$, we have

$$\begin{aligned}
&\|\vartheta(t)-\vartheta^*(t)\|\\
&\quad=\|(\Sigma\vartheta)(t)-(\Sigma\vartheta^*)(t)\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left\| k(t,\vartheta_{\rho(t,\alpha_t+\vartheta_t)}+\alpha_{\rho(t,\alpha_t+\vartheta_t)}) - k(t,\vartheta^*_{\rho(t,\alpha_t+\vartheta_t)}+\alpha_{\rho(t,\alpha_t+\vartheta_t)}) \right\| \\
&\quad + \int_0^t \|Q(t-s)\|_{\mathcal{L}(X)} \left\| h(s,\vartheta_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}) - h(s,\vartheta^*_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}) \right\| ds \\
&\leqslant \pi(t) \left(\|\vartheta_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}} + \|\vartheta^*_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}} \right) \\
&\quad + M\int_0^t e^{-\gamma(t-s)} f(s) \left(\|\vartheta_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}} + \|\vartheta^*_{\rho(s,\alpha_s+\vartheta_s)}+\alpha_{\rho(s,\alpha_s+\vartheta_s)}\|_{\mathcal{B}} \right) ds \\
&\leqslant 2\kappa(\Xi+\mu)\pi(t) + 2M(\kappa\Xi+\kappa\mu)\int_0^t e^{-\gamma(t-s)} f(s) ds.
\end{aligned}
$$

Using (H3) and (4.1), we conclude that for two solutions $\vartheta,\ \tilde{\vartheta}$ elements of the closed ball $\bar{B}(\vartheta^\star,\Xi)$ we have

$$\lim_{t\to\infty} \|\vartheta(t)-\tilde{\vartheta}(t)\| = 0.$$

This implies the local attractivity of the solutions of system (1.1).

5. Application

This section concerns the illustration of our results. So, we consider the system

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\sigma(t,\tau) + \int_{-\infty}^t a(t-s)\sigma(s-\rho_1(t)\rho_2(\|\sigma(t,\tau)\|),\tau) ds \right) \\
&\quad = \frac{\partial^2}{\partial \tau^2}\left(\sigma(t,\tau) + \int_{-\infty}^t a(t-s)\sigma(s-\rho_1(t)\rho_2(\|\sigma(t,\tau)\|),\tau) ds \right) \\
&\quad + \int_0^t b(t-s)\left(\sigma(s,\tau) + \int_{-\infty}^s a(t-l)\sigma(l-\rho_1(t)\rho_2(\|\sigma(t,\tau)\|),\tau) dl \right) ds \\
&\quad + \int_{-\infty}^t c(s-t)\sigma(t-\rho_1(t)\rho(\|\sigma(t,\tau)\|,\tau) ds, \\
&\quad \begin{cases} \sigma(t,0)=\sigma(t,\pi)=0, & t\geqslant 0, \\ \sigma(s,\tau)=\sigma_0(s,\tau), & s\in(-\infty,0],\ \tau\in[0,\pi], \end{cases}
\end{aligned}
\tag{5.1}
$$

where $b : \mathbb{R}^+ \to \mathbb{R}$, $\rho_1,\ \rho_2 : J \to \mathbb{R}_+$ and $a,\ c : \mathbb{R}\to\mathbb{R}$ are continuous functions.

Let $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ be the operator defined as follows $Au = u''$ with domain $D(A) = \{u \in X : u'' \in X,\ u(0) = u(\pi) = 0\}$ which generates a C_0-semigroup on X.

For $t \geqslant 0$ and $y \in D(A)$, we consider the operator $\Upsilon(t)$ defined by

$$\Upsilon(t) : D(A) \to X, \qquad y \to b(t)Ay.$$

Next, we consider the phase space introduced by Hino et al. (see [16]) given by $\mathcal{B} = C_0 \times L^p(\tilde{h}, X)$, where $\tilde{h} : (-\infty, 0] \to \mathbb{R}$ is non-negative measurable function satisfying the appropriate conditions in Definition 2.4.

For rewriting the problem (5.1) in an abstract form in X, we give the following notations

$$
\begin{cases} x(t)(\tau) = \sigma(t,\tau) & \text{for } t\geqslant 0 \text{ and } \tau\in[0,\pi], \\ \phi(t)(\tau) = \sigma_0(t,\tau) & \text{for } t\leqslant 0 \text{ and } \tau\in[0,\pi]. \end{cases}
\tag{5.2}
$$

Let us introduce the operators $k,\ h : J\times\mathcal{B}\to X$ by

$$k(t,y_t)(\tau) = \int_{-\infty}^t a(t-s)y(s-\rho(t,y_t))(\tau) ds, \qquad h(t,y_t)(\tau) = \int_{-\infty}^t c(t-s)y(s-\rho(t,y_t))(\tau) ds,$$

with $\rho : J \times \mathcal{B} \to \mathbb{R}$ defined by

$$\rho(t, y_t) := \rho_1(t)\rho_2(\|y(t)(\tau)\|).$$

Then for $\phi \in \mathcal{B}$, problem (5.1) takes the following abstract form

$$\begin{cases} \frac{d}{dt}\left[x(t) - k(t, x_{\rho(t,x_t)})\right] \\ \quad = A[x(t) - k(t, x_{\rho(t,x_t)})] + \int_0^t \Upsilon(t-s)\left[x(s) - k(s, x_{\rho(s,x_s)})\right] ds + h(t, x_{\rho(t,x_t)}), & t \in J := [0, \infty), \\ x(t) = \phi(t), & t \in (-\infty, 0]. \end{cases}$$

We suppose that b is bounded and C^1-function such that b' is bounded and uniformly continuous, then (C1) and (C2) hold. Consequently, by Theorem 2.2, there is a resolvent operator $(Q(t))_{t \geqslant 0}$ on X for Eq. (2.1). Moreover, we assume that there exists $\beta > a > 1$ and $b(t) < \frac{1}{a}e^{-\beta t}$ for all $t \geqslant 0$. Then, the problem (5.1) admits a mild solution which is locally attractive.

Acknowledgments

The authors would like to thank the editor and the reviewers for their constructive comments and suggestions which improved the quality of the paper. This work was supported by CEA-MITIC (Universit Gaston Berger) and CEA-SMA (Bénin), the Réseau EDP-Modélisation et Contrle.

References

[1] K. Balachandran, R. R. Kumar, *Existence of solutions of integrodifferential evolution equations with time varying delays*, Appl. Math. E-Notes, **7** (2007), 1–8. 1
[2] J. Banaś, B. C. Dhage, *Global asymptotic stability of solutions of a functional integral equation*, Nonlinear Anal., **69** (2008), 1945–1952. 1
[3] J. Banaś, B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl., **284** (2003), 165–173.
[4] J. Banaś, T. Zając, *Solvability of a functional integral equation of fractional order in the class of functions having limits at infinity*, Nonlinear Anal., **71** (2009), 5491–5500. 1
[5] T. A. Burton, C. Kirk, *A fixed point theorem of Krasnoselski–Schaefer type*, Math. Nachr., **189** (1998), 23–31. 2.7
[6] C. Corduneanu, *Integral equations and stability of feedback systems*, Academic Press, New York-London, (1973). 2.6
[7] K. Ezzinbi, S. Ghnimi, *Existence and regularity of solutions for neutral partial functional integrodifferential equations*, Nonlinear Anal. Real World Appl., **11** (2010), 2335–2344. 1
[8] K. Ezzinbi, S. Ghnimi, M. A. Taoudi, *Existence and regularity of solutions for neutral partial functional integrodifferential equations with infinite delay*, Nonlinear Anal. Hybrid Syst., **4** (2010), 54–64.
[9] K. Ezzinbi, H. Toure, I. Zabsonre, *Local existence and regularity of solutions for some partial functional integrodifferential equations with infinite delay in Banach space*, Nonlinear Anal., **70** (2009), 3378–3389.
[10] Y. Fujita, *Integrodifferential equation which interpolates the heat equation and tne wave equation*, Osaka J. Math., **27** (1990), 309–321. 1
[11] R. C. Grimmer, *Resolvent operators for integral equations in a Banach space*, Trans. Amer. Math. Soc., **273** (1982), 333–349. 2.1, 2.2
[12] J. K. Hale, J. J. Kato, *Phase spaces for retarded equations with infinite delay*, Funkcial. Ekvac., **21** (1978), 11–41. 1, 2
[13] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, (1993). 1
[14] F. Hartung, J. Turi, *Identification of parameters in delay equations with state–dependent delays*, Nonlinear Anal., **29** (1997), 1303–1318. 1
[15] E. Hernández M., M. A. McKibben, *On state–dependent delay partial neutral functional–differential equations*, Appl. Math. Comput., **186** (2007), 294–301. 3.3
[16] Y. Hino, S. Murakami, T. Naito, *Functional differential equations with Infinite delay*, Springer-Verlag, Berlin, (1991). 1, 2, 3.2, 5
[17] J. Liang, J. H. Liu, T.-J. Xiao, *Nonlocal impulsive problems for nonlinear differential equations in Banach spaces*, Math. Comput. Modelling, **49** (2009), 798–804. 2.3
[18] A. M. Lyapunov, *The general problem of the stability of Motion*, (Translated by A. T. Fuller from Edouard Davaux's French translation (1907) of the 1892 Russian original), Internat. J. Control, **55** (1992), 531–534. 1
[19] N. I. Mahmudov, *Existence and uniqueness results for neutral SDEs in Hilbert spaces*, Stoch. Anal. Appl., **24** (2006), 79–95. 1